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The Effects of the Unit Concept on Prospective Elementary Teachers' Understanding of Rational Number Concepts.

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THE EFFECTS OF THE UNIT CONCEPT ON PROSPECTIVE ELEMENTARY TEACHERS' UNDERSTANDING OF RATIONAL NUMBER CONCEPTS

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ABSTRACT

The purpose of this study was to follow and describe the cognitive processes of five prospective elementary teachers as they engaged in the formation of units and to examine the role of the unit concept as a possible link between the whole number and rational number domains. An attempt was made to gain an understanding of how the students constructed units and whether or not their attention to and understanding of the unit concept would increase their understanding of rational number concepts and operations.

The rational number domain is one that causes great difficulties for students and their teachers. The complexity of this domain is revealed through the many roles in which a rational number can appear - measure, ratio, part-whole, quotient, and operator. In an effort to improve rational number understanding, focus has turned to the unit fraction and the basic concept of unit.

It has been suggested that students possess intuitive or informal knowledge of unit formation and this knowledge may be used as a foundation for building rational number understanding. This study examined the role of the unit concept in bridging the gap between whole numbers and rational numbers.

The students were five preservice elementary teachers enrolled in a mathematics course designed for elementary education majors. The group of five students was selected based on an inventory and personal interviews. Once selected the students participated in a teaching experiment that consisted of six lessons. Data was collected through videorecording, audiotapes, journals, essays, and students' written work.

Results of the study indicated: (a) Students' awareness of their informal knowledge regarding the unit concept promotes understanding; (b) teachers who provide opportunities for students to build on their informal knowledge by working with various whole number units to develop unitizing and norming skills, help students develop schemes for further work with rational numbers; and (c) students who become accustomed to focusing on the unit may more readily recognize intuitive and authentic connections between natural and rational numbers.

CHAPTER ONE

INTRODUCTION

The intent of this first chapter is to justify the undertaking of the present study by framing it within the larger field of mathematical didactics. The chapter is divided into four sections. The first section provides a statement of the problem along with supporting documentation. In the second section, the possible causes of the problem are presented. The attempts at finding solutions for the problem are discussed in the third section. A series of research questions arising from issues raised in the first three sections, along with the organization of the study, constitute the fourth section.

The Problem

The problem is simple: Students, and many of their teachers, simply do not understand the concepts of rational numbers. This struggle with rational numbers has been well documented (Behr, Lesh, Post, & Silver, 1983; Kieren, 1976; Post, Harel, Behr, & Lesh, 1988). Students fail to "internalize a workable concept of rational number" (Behr, Wachsmuth, Post, & Lesh, 1984, p. 323) and therefore their overall performance with rational numbers has been poor. This poor performance may be a direct result of inadequate conceptual understanding on the part of the teacher. Recent studies reveal that many prospective teachers know what good mathematics teaching should involve, but they are limited by their conceptual understanding of the topic to be presented (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992). One area in which preservice, as well as in-service, teachers' have demonstrated a frightening lack of conceptual knowledge is that of rational numbers (Post et al., 1988; Thipkong & Davis, 1991). As noted by Thipkong and Davis (1991) "it is important to know preservice teachers' weaknesses in order to help them become better in their subject matter in preparation for teaching students since today's preservice teachers are tomorrow's teachers" (p. 93).

The Cause

While the cause of the lack of conceptual understanding in the domain of rational numbers can not be attributed to a single source, the curriculum has played a major role. The lack of conceptual knowledge of our teachers has resulted in their delivering a curriculum which emphasizes procedures rather than understanding (Behr et al., 1983). Students have memorized the algorithms, often incorrectly, but have no knowledge of the concepts underlying the procedures (Mack, 1990). Difficulties with rational numbers are heightened by misconceptions that arise as students try to give meaning to the teacher-taught algorithms (Fischbein, Deri, Nello, & Marino, 1985; Mack, 1990). As students become exposed to the notion of rational numbers they attempt to find a connection with something already familiar, like whole numbers. They try to fit the new idea of rational numbers into their existing schema and frames of whole numbers. When a natural connection is not identified, misconceptions occur as the new knowledge is forced to conform to pre-existing schema (Skemp, 1987). One of the more common misconceptions that surface in operations of rational numbers is the student generated strategy referred to as whole number dominance (Behr et al., 1984). An example of this strategy is when students attempt to add rationals by adding the numerators and adding the denominators. Studies by Graeber, Tirosh, and Glover (1989) indicate that the misconceptions established by children are not outgrown. Many of our preservice teachers possess the same misconceptions.

A Possible Solution

Some researchers contend that students' informal knowledge can provide a base for developing an understanding of rational number (Lamon, 1992; Mack, 1990). Researchers have argued that much of what we know has been learned outside of, or in spite of school instruction (Lave, 1988; Mack, 1990). Mack (1990) refers to this type of knowledge as informal knowledge. She defines informal knowledge as "knowledge related to real-life situations that students construct and bring to instruction..." (p. 16) and contend that "both children and adults possess a rich store of informal knowledge" (p. 16). This informal knowledge may be either correct or incorrect and has a direct influence on our performance in a variety of mathematical content areas. Work by Fischbein et al. (1985) and Mack (1990) contends that misconceptions can arise from this primitive or informal knowledge but this type of knowledge can also be used as a foundation on which to build meaningful procedures (Mack, 1990; Pothier & Sawada, 1983).

Unit Formation

It has been suggested that the formation of units is informal knowledge that can aid rational number understanding (Lamon, 1992). A rational number can be defined as any number of the form m/n where m and n are integers and n is not equal to zero. Children's initial understanding of rational numbers is not derived from m and n, but rather from physical embodiments (Post, Wachsmuth, Lesh, and Behr, 1985). These embodiments might be a picture of a pie cut into n equal pieces with m of them shaded or a set of n circles with m of them shaded. In any case, the embodiment involves a partitioning or "fracturing" (Freudenthal, 1983) of some physical or mental object. This object is a unit.

A unit may contain one object, a group of objects, or may itself be composed of units. For example, consider 12 balls. There are numerous unit interpretations of this, among them are the following: (a) Considering each of the 12 balls as single units, you have 12 one-units. (b) A partitioning of these 12 balls into subgroups would provide the basis for forming composite units or units of units. For example, two six-units, or three four-units. (c) A unit of units of units could be created by further partitioning the two six-units. For example, two groups of three two-units. The unit interpretation of 12 balls demonstrates that an understanding of the unit concept involves viewing a whole as a nested system of units.

Von Glasersfeld (1981) suggests an intuitive aspect in his discussion of unit. He describes units as conceptual structures determined by "focused and unfocused attentional pulses" (Von Glasersfeld, 1981, p. 87). "A group of co-occurring sensory-motor signals becomes a 'whole' or 'thing' or 'object' when an unbroken sequence of attentional pulses is focused on these signals and the sequence is framed or bounded by an unfocused pulse at both ends" (Von Glasersfeld, 1981, p. 87). To illustrate the relationship between the focus and the unit consider a child beginning to count the fingers on one hand. The focus is on individual fingers. Each finger represents a unit. The five fingers are considered five one-units. After a while the child associates the counting number five for the fingers on one hand. The focus is no longer on the individual fingers but rather on a "handful" of fingers. The units of five singleton units have been transformed to one five-unit. This is a more sophisticated way of thinking and illustrates the child's natural or intuitive ability to form units.

Mathematics of quantity.

Work by Behr, Harel, Post, and Lesh (1992) has focused on the unit. As children deal with whole numbers, most traditional problems focus on units of one. For example, consider the following problem: Stephen has two bags with four marbles in each bag. Josh gives him three more bags with six marbles in each bag. How many bags with two marbles can they make? The traditional approach to this problem suggests a solution of 2x4=8, 3x6=18, 8+18=26, 26÷2=13. This approach emphasizes what is called a mathematics of number as compared to a mathematics of quantity. In this solution all units are changed to units of size one. A more natural approach used by children when asked to act-out the problem is to go directly to units of two: two four-unit sacks equals two groups of two two-unit sacks which is the same as four two-unit sacks; three six-unit sacks equals three groups of three two-unit sacks which is the same as nine two-unit sacks; four two-unit sacks plus nine two-unit sacks equals 13 two-unit sacks. In this approach the focus is on the quantities, consisting of a number and a unit, not just on the numbers. This emphasizes a mathematics of quantity.

Unitizing and norming.

The basic issue involved in the whole number problem above is reconceptualizing the situation in terms of a fixed unit or standard. Freudenthal (1983) refers to this process as norming. Traditionally the first time students encounter a situation in which this notion of norming is critical is in the domain of rational numbers. In an addition such as one-third plus one-half, finding a common denominator (or norm) is helpful. In an addition such as two-thirds plus four-fifths finding a common denominator is nearly essential to find an exact sum. In the case of one-third plus one-half the process of finding a common denominator involves the reconceptualization of one-third as two-sixths and one-half as three-sixths. The next step in the traditional algorithm is merely to add the numerators and "bring over" the denominator. Research by Lamon (1992) suggests a more conceptual approach through the notion of unitizing. Unitizing refers to the formation of composite units. That is, the ability to recognize two-sixths plus three-sixths as 2(1/6-unit)s plus 3(1/6-unit)s. By focusing on the unit, addition and subtraction of rational numbers is merely an extension of addition and subtraction of whole numbers. This provides a natural connection between the whole number domain and the rational number domain.

The Current Study

The need to connect the ideas and procedures among mathematical domains is reflected in the National Council of Teachers of Mathematics' (NCTM) Curriculum and Evaluation Standards for School Mathematics (1989). This document voices the sentiments of the current national reforms in mathematics education which stress the importance of teaching for connections. Without these connections students' knowledge of mathematics is dependent upon the memorization of a set of unrelated rules and procedures. This study examined the potential of the unit concept as one such connector by investigating the conceptualization of the unit as a way to bridge the gap between whole numbers and rational numbers. By building on formation of units as informal knowledge, a natural connection can be established between whole numbers and rational numbers, thereby aiding in the conceptual understanding of and reducing common misconceptions associated with rational numbers.

<u>Research Questions</u>

I did not have specific testable hypotheses, but rather I was trying to gain an understanding of how the students' construct units and whether or not their specific attention to the unit concept would increase their understanding of rational number concepts and operations. The following questions were considered:

1. Do preservice elementary teachers exhibit informal knowledge regarding unit formation?
2. Are they aware of their own construction of units?
3. What cognitive obstacles are encountered in the process of understanding the unit concept?
4. How will an awareness of the informal nature of the unit concept affect their problem solving performance on whole number addition and subtraction?
5. Will knowledge of the role of the unit concept in the whole number domain facilitate learning of concepts in the domain of rational numbers?

The research questions were addressed through a variety of qualitative data collected during the teaching experiment.

<u>Organization of the Dissertation</u>

A review of the literature is provided in Chapter 2. The chapter begins with a general overview of rational numbers and the difficulties encountered by students and their teachers in the domain of rational numbers. Then there is a discussion of the unit fraction and its role in the rational number dilemma. Finally there is a review of the literature on the unit concept and its informal nature.

In Chapter 3 a discussion of the qualitative nature of the study and a justification of the selection of the teaching experiment methodology is given. The teaching experiment focused on the cognitive processes of five prospective elementary teachers as they engaged in lessons involving unit formations and transformations. Data were collected from a variety of sources in an attempt to better understand the students' cognitive reactions to the unit concept.

Chapter 4 provides the results of each lesson and an analysis of the individual responses. Chapter 5 provides an analysis of the qualitative data collected through the videorecordings, interviews, journals, etc. as it relates to each of the research questions. Chapter 6 summarizes the study and discusses the results. The implications for future practice and research are also presented in chapter 6.

CHAPTER TWO

REVIEW OF THE LITERATURE

The literature review presented in this chapter, while not exhaustive, provides a setting in which to situate the present study. As presented in chapter 1, my contention is that the formation of units is an intuitive or informal ability that can be expanded to reveal a natural connection between whole number and rational number concepts and operations.

This review is divided into three sections. The first section presents a general overview of the literature about rational number concepts. The second section focuses more closely on studies involving the unit fraction (i.e. fractions of the form 1/n, where n is a non-zero whole number). These studies reveal the necessity of starting with a basic concept on which to build rational number knowledge. If the basis of rationals is the unit fraction, then the foundation for thinking in terms of unit fractions must be the notion of a unit. The third section presents research on the unit concept and situates the unit concept in terms of informal knowledge.

Rational Numbers

Children encounter numerous sets of numbers throughout their school math experience. The first set to which they are exposed is called the natural or counting number (i.e., 1,2,3,...). These numbers are normally used to measure (count) discrete quantities. For example, the number of candies in a bag of candy or the number of cards in a pack of baseball cards. An empty candy bag suggests an extension of the natural numbers to include the notion of zero. The set of natural numbers united with the zero form what is called the set of whole numbers (i.e., 0,1,2,3,...).

The integers are an extension of the whole numbers in that they include the set of whole numbers as well as their opposites (i.e.,...-3, -2, -1, 0, 1, 2, 3,...). These numbers can be used to measure a discrete quantity or to measure a deficit of quantity. For example, -5 may indicate that five dollars is missing from an account.

While the set of integers provides a means of measuring discrete or finite collections of objects, the needs of our daily life often call for the measurement of various quantities such as length, weight, and time. Very seldom does a given length contain an exact integral number of linear units. To satisfy these measuring tasks we need the set of rational numbers. A rational number is commonly defined as the quotient of two integers, a/b, $b \neq 0$ (e.g., -1/2, 2/3, 3/4, etc.). Within the system of rational numbers, the measuring of discrete and continuous quantities are possible (Eves, 1953).

The above discussion reveals that the rational numbers are an extension of the integers, which, in turn, are an extension of the whole numbers. Since the very origin of rational numbers is connected to the set of whole numbers, it only seems fitting that the operations within these sets, mirror that same connection.

Throughout this study you will see references to fractions along with rational numbers. It is customary to refer to the positive rational numbers as fractions (Freudenthal, 1983).

Difficulties with Rational Numbers

The difficulties children encounter with rational numbers has been well documented (Behr et al., 1983; Hart, 1988; Heller, Post, Behr, & Lesh, 1990; Kieren, 1976; Mack 1990; Van den Brink & Streefland, 1979). A recent National Assessment of Educational Progress in the United States (NAEP) indicate that students calculate fractions by applying memorized algorithms and demonstrate little or no knowledge of the underlying concepts. Work by Behr et al. (1983) attribute these difficulties to a curriculum which has emphasized procedures rather than understanding. According to Mack (1990) "...many students' understanding of fractions is characterized by a knowledge of rote procedures, which are often incorrect, rather than by the concepts underlying the procedures" (p. 17).

Post et al. (1988) suggest that the difficulties with rational numbers are not limited to elementary and middle grades students. Many of the same misunderstandings of students are shared by their teachers. A study conducted by the Rational Number Project (RNP) attempted to develop a model middle school mathematics teacher education program. The objective of the RNP study with middle school teachers was to generate a profile of the mathematical understanding of teachers. For as Post et al. (1988) note, "we really do not know very much about what mathematics intermediate level (grades 4-6) teachers actually do know and understand" (p. 200).

The study included 218 middle school teachers (167 from Minnesota and 51 from Illinois). The assessment instrument used to assess the teachers had three parts - short answer, pedagogical explanations of solutions, and a two hour interview. The intent of the instrument was to try to understand the way teachers understand important concepts, not to evaluate what they did or did not know. Items were designed to reflect the conceptual underpinnings of rational number topics for grades 4, 5, and 6. Some of the items related to rational numbers included; part-whole, ordering fractions, fraction equivalence, concept of unit, and operations with fractions.

The results of the assessment were very disturbing (see Table 1). "Many teachers simply do not know enough mathematics" (Post et al., 1988, p. 213) to promote conceptual understanding in their students. Some of the fundamental items of the test were missed by almost half of the teachers and only a small number of teachers who could correctly solve the problems were able to explain their solutions in a pedagogically acceptable manner.

Table 1

Results of RNP Project with Middle School Teachers

Topic	Number of items	% correct
Part-whole	2	68%
Ordering fractions	4	68%
Fraction Equivalence	4	50%
Concept of Unit	4	69%
Operations/fractions	13	72%

No wonder our students have difficulties. Approximately 30% of the teachers had trouble with the questions regarding rational numbers. As Post et al. (1988) note,

> We fail to understand how teachers without a relatively firm foundation could possible be in a position to present and explain properly, to ask the right question at the right time, and to recognize and encourage high levels of student mathematical thinking when it occurs. (p. 214)

The lack of conceptual knowledge demonstrated by our prospective elementary and middle grade teachers is not limited to rational numbers. Work by Simon and Blume (1992) has examined the conceptual knowledge of prospective teachers in regard to area. Their study with 26 preservice teachers revealed that these students "do not have a well-developed concept of area nor an understanding of why the relationship of the length and width of a rectangle to its area is appropriately modeled by multiplication" (p. 27). These prospective teachers had memorized the traditional formula but had no conceptual understanding of the connection between area and multiplication.

Graeber et al. (1989) revealed teachers' misconceptions in solving verbal problems in multiplication and division. The subjects of their study were 129 female college students enrolled in a mathematics course for early elementary education majors. An 18-item test was administered in which the subjects were asked to write an expression for solving the problem, but they did not have to perform the operation.

Results of the study indicated that 39% of the preservice teachers answered four or more of the 13 multiplication or division problems incorrectly. Personal interviews conducted with the subjects reveals that every subject "gave evidence of holding at least one of the misconceptions" (p. 97). The common misconceptions were (a) you must always divide by a whole number, (b) multiplication always makes bigger, and (c) division always makes smaller.

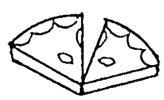
Subconstructs

The lack of conceptual knowledge of rational numbers demonstrated by teachers and students may be attributed to the complexity of this domain. Post, Behr, & Lesh (1986) suggest that rational number concepts involve the coordination of several variables. The variables involved can be thought of as interpretations, personalities, or subconstructs (Kieren, 1976; Behr et al., 1984; Post et al., 1986). A complete understanding of rational numbers requires an understanding of each subconstruct as well as an understanding of how the subconstructs are interrelated (Behr et al., 1983; Freudenthal, 1983; Kieren, 1976; Vergnaud, 1988). While Kieren (1976) and Behr et al. (1983) differ slightly in their identification of the subconstructs, their works can best be summarized by the subconstructs of ratio, part-whole, measure, quotient, and operator.

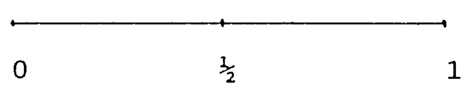
In an attempt to develop the meaning of these subconstructs consider the fraction one-half. From the ratio perspective, one-half could be illustrated by the statement "one of every two students is female":

♀♂ ♀♂ ♀♂ ♀♂ ♀♂ . . .

As a part-whole interpretation one-half is represented as one piece of a pie cut into two equal pieces:



The fraction one-half may be represented as a measure by a position on the number line:



The quotient subconstruct of rational numbers involves focusing on one-half as one divided by two. In other words, one-half can represent the amount of a cookie each person eats if one cookie is shared by two people or the amount if two cookies are shared by four people, etc.

The operator subconstruct of a rational number suggests that one-half can be thought of as a function which is applied to some number, object, or set (Behr, Harel, Post, & Lesh, 1992).

5 $\boxed{\tfrac{1}{2} \text{ of}}$ $\frac{5}{2}$ □ $\boxed{\tfrac{1}{2} \text{ of}}$ ▯   ** ** $\boxed{\tfrac{1}{2} \text{ of}}$ * *

Behr et al. (1983) claim that the subconstruct involving the part-whole relationship is fundamental in developing the other personalities of rational numbers. The doctoral dissertation of Boulet (1993) argues that the part-whole relationship is the ratio between part and whole, therefore ratio is the first and most basic of the subconstructs. In order to understand the ratio between a part and the whole, one should begin with the simplest case - the unit fraction.

Unit Fractions

Historically, the concept of unit fraction preceded the concept of fraction in general. A unit fraction is any fraction having a numerator of one and any other natural number as a denominator (i.e. 1/n). The Egyptians represented all fractions, except two-thirds, as the sum of unit fractions to "avoid some of the computational difficulties" (Eves, 1953, p. 41). For example, 2/7 was expressed as 1/4 plus 1/28 and 2/99 as 1/66 plus 1/198. Mathematical tables of the fraction forms offered only one decomposition for a particular fraction (Eves, 1953). The notation of the unit fraction consisted of an elliptical symbol placed above the denominator number. The fraction two-thirds had its own symbols as noted below.

1/3 = $\overset{\circ}{3}$ and 2/3 = 

Work by Saenz-Ludlow (1993), Olive (1993) and Steffe and Spangler (1993) at the University of Georgia has addressed the importance of the unit fraction in developing fraction schemes. A series of computer microworlds have been developed to observe children's fraction schemes. Through the use of a mouse children are able to cut a given stick (unit) into pieces or cut off pieces from the stick that can be replicated to serve as new units. The cutting process can be repeated with the new unit. This environment promotes the recursive construction of iterable fractional units. A fraction is iterable when the child realizes that multiple repetitions of the iterated fraction result in a new composite unit. But this realization must occur before the repetitions are made. The ability to use iterable fractional units enables the child to model any common fraction by iterating a unit fraction, for example, three-fourths as three iterations of one-fourth (i.e., 3/4 = 1/4 + 1/4 + 1/4). This development is necessary for a quantitative understanding of fractions.

Quantitative Understanding

The quantitative notion of fractions is concerned with how fractions are perceived as numbers (Post et al., 1986). It involves a complex network of interrelated subconcepts. Some of the subconcepts include (a) realization that rational numbers are numbers, (b) rationals can be expressed in many different forms, (c) rationals have relative and absolute sizes and can be ordered, (d) relationship between the numerator and denominator.

Saenz-Ludlow (1994) contends that the student must conceptualize fractions as quantities "before they are introduced to standard symbolic computational algorithms" (p. 50). She conducted a year long teaching experiment to analyze the fraction schemes of six third graders. The goal of the study was "to explore children's efforts to use their natural-number knowledge to generate their initial conceptualization of fractions" (Saenz-Ludlow, 1994, p. 51). Each child participated in four clinical interviews and fifteen teaching episodes. Tasks for the study were designed to encourage the students to use their natural-number units to create fractional units. The analysis of the study revealed that the students' quantitative reasoning with fractions was based on their quantitative reasoning with natural numbers. The conceptualization of a fraction as a unit evolves from the act of physical or mental measuring. Measuring requires an awareness of the whole to be measured, the selection of the "ruler" or measuring unit, and "a segmentation of the unity of the whole by partition" (Saenz-Ludlow, 1994, p. 81). Once measured the whole suggests either a part-to-whole or whole-to-part operation. When a whole (e.g., unit A) is measured by a smaller unit (e.g., unit B) the whole is conceptualized as a composite unit and a one-to-many or a part-to-whole relation is established (i.e., Unit A = 4(B unit)s). When the smaller unit (Unit B) is conceptualized as a fractional part of the whole, the whole-to-part relation is established (i.e., Unit B = 1/4(Unit A)). Saenz-Ludlow (1994) contends that fractional quantification results from the concurrent establishment of these two operations. Saenz-Ludlow (1994) concludes by stating, "helping children to conceptualize natural-number units seems to be necessary spadework for the teaching of fractions" (p. 83).

Behr et al. (1984) conducted an 18-week teaching experiment which provided data concerning the thought processes or solution strategies used by children in dealing with order and equivalence. Post et al. (1986) contend that order and equivalence is a very important part of quantitative understanding. This is consistent with Schwartz' (1988) notion that ordering "constitutes a reasonable probe of competent understanding of intensive quantity" (p. 43).

The students in the teaching experiment (Behr et al., 1984) were given three classes of problems; fractions with the same numerators, fractions with the same denominators, and fractions with different numerators and denominators. The analysis of the problems revealed several distinct strategies generated by the children. Upon comparing the thinking strategies used on the three classes of fractions, four strategies were common to all three classes: (a) thinking that involved attention to both the numerator and the denominator, (b) thinking that depends on manipulatives, (c) thinking that referred to a third fraction when comparing two fractions (reference-point strategy), and (d) thinking influenced by one's knowledge of whole numbers (whole-number dominance strategy). The tendency for students to rely on whole number skills is supported by Kieren (1993). He contends that while it is typical in mathematics for students to see fractions as extensions of whole numbers, simply extending the rules of whole numbers is not appropriate. The development of the unit concept as proposed in this study would support the dependence on whole number skills by revealing rationals as a natural extension of the whole numbers but will also mitigate against a simply use of extended rules by emphasizing the unit.

From this perspective, that the basis of fraction understanding is the unit fraction, it follows the informal knowledge of unit formation is an appropriate knowledge base for developing rational number concepts and operations.

The Unit Concept

Attention to the concept of unit is not new. Many researchers have addressed the importance of the unit concept in the understanding of number (Piaget, Inhelder, & Szeminska, 1960; McLellan & Dewey, 1895; von Glasersfeld, 1981). Hunting and Sharpley (1988) contend that many of the concepts and procedures taught in school mathematics are based on units, but only those units "that form the core of whole number arithmetic (ones, tens, hundreds, etc.)" (p. 175). The concept of unit referenced in this study pertains to the view expressed by Galperin and Georgiev (1969) "...all elementary mathematical concepts, regardless of the limitations of their content, assume the notion of unit" (p.1).

Mathematics of Quantity

Work by Behr, Khoury, Harel, Post, and Lesh (1992) and Schwartz (1988) has indicated the importance of the unit concept. The work by Behr, Harel, Post and Lesh (1992) has focused on viewing rationals from the perspective of mathematics of quantity. The mathematics of quantity approach proposes that the "units of measure and the magnitude of quantities are both significant to the understanding of number relations and operations" (Behr, Harel, Post, and Lesh, 1992, p. 23). As mentioned in chapter 1, the mathematics of quantity focuses on natural unit formation as opposed to the mathematics of number approach, the more traditional approach, which tacitly assumes a unit of one. This traditional approach has resulted in an apparent disregard for units that can be seen in traditional classrooms. When solving application problems final answers are frequently reported in numerical form with no reference to the unit (e.g., 7 instead of 7 inches, 5 instead of 5 mph, etc.) In problems containing different units students often disregard the distinction and simple operate on the numbers (e.g., Josh can peddle at 8 mph on his bike. After 45 minutes how far has he traveled? Answer is given as 360). This lack of attention to units, or inattention to the quantities in a problem and relationships among them, leaves problem solvers vulnerable to such misconceptions as division always makes smaller and multiplication always makes bigger.

Schwartz.

When two mathematical quantities are composed they yield a third quantity which either preserves the original referents or transforms them. Schwartz (1988) refers to referent preserving compositions and referent transforming compositions and contends that a new approach to teaching and learning mathematics should be based on distinguishing between these two compositions.

Addition and subtraction are referent preserving compositions of quantity since the quantity produced is like the original referents (e.g., 2 ft + 3 ft = 5 ft). Multiplication and division are referent transforming compositions of quantity since the quantity produced is not like either of the two original quantities (e.g. 5 lbs x 3.00 dollars/lb = 15.00 dollars).

In a referent transforming composition one needs to distinguish between two different kinds of quantity; extensive quantity and intensive quantity. The intensive quantity is a relationship between two, usually extensive, quantities. In the example given above lbs and dollars are extensive quantities, while dollars/lb is an intensive quantity. Schwartz (1988) contends that an introduction of intensive quantity "is essential to understanding the vast majority of situations that call for the arithmetic acts of multiplication and division" (p. 46). He also contends that the idea of referent transforming composition which distinguishes between intensive and extensive quantities will not only improve the future understanding of multiplicative structures and rational numbers but also "offers an opportunity to repair a substantial amount of poorly taught and poorly learned mathematics" (p. 42).

Behr.

An analysis of rational numbers with emphasis on the mathematics of quantity has been conducted by Behr, Harel, Post, and Lesh (1992). In this analysis two notational systems were developed to exhibit unit formation and transformations. These notations, the "bridging notation" and the "mathematics of quantity", were used to provide a content/semantic analysis of the rational numbers. The bridging notation was used to provide "a generic noncontextualized pictorial system representing the manipulation of objects at the concrete level" (Behr, Harel, Post, and Lesh, 1992, p. 301). It involves symbols like 0, *, #, etc. each of which is used to represent an object. Enclosure of these within parentheses, brackets, or braces is used to denote a conceptualization of the objects as units. Units can be singleton (one object), composite (more than one object) or intensive (measure unit). The representation for a three block unit might be (0 0 0).

The mathematics of quantity notation provides a correspondingly more formal representation. This notation involves using abstract rather than specific unit labels. For example, instead of writing a unit as six balls we could call this a six-unit and denote it as 1(6-unit).

The combined analysis based on the two notational systems provides an extensive content/semantic analysis of rational numbers. According to Behr, Harel, Post, and Lesh (1992) the analysis of the different rational number constructs suggests that understanding of these constructs depends on formation and transformations of unit structures consistent with a set of conversion principles as well as rather deep knowledge about concepts of measurement.

In a study with 30 preservice elementary school teachers focusing on the operator construct of rational numbers Behr, Khoury, Harel, Post, & Lesh (1992) examined the students conceptualization of the unit. Students were asked to solve problems involving bundles of sticks. Each bundle contained four sticks and was secured by a rubber band. Students were shown that the rubber band could be removed and replaced if desired. Students were given a pile of sticks consisting of eight bundles of four sticks and asked to show three-fourths as many sticks.

Evidence of unit reformation was indicated as students removed rubber bands thereby changing the size of the unit (size-exchange strategy) or re-grouped the bundles thereby changing the number of units (number-exchange strategy). In other words, the students using the size-exchange strategy took three of the four sticks in each bundle to form eight bundles with three sticks each (24 sticks). The Number-exchange strategy involved grouping the eight bundles into four groups (two bundles per group) and then taking three of the four groups. This process resulted in six bundles of four sticks (24 sticks). While both strategies resulted in the same number of sticks, the two strategies reveal a different conceptualization of units.

The works by Schwartz (1988) and Behr, Harel, Post, and Lesh (1992) described above indicate the importance of the concept of unit, but how does this begin? Von Glasersfeld (1981) suggests that the foundation for the unit concept develops early and in out-of-school situations. He contends that the basic notion of the unit concept is intuitive.

> We do divide our visual, auditory, and tactual fields of experience into separate parts which, in our cognitive organization, then become individual items or "things." That is to say, we quite successfully differentiate or "cut" things out of a background and perceive each one of them as an entity or whole. (p. 86)

Informal Knowledge

Many researchers have argued that much of what we know has been learned outside of, or in spite of school instruction (e.g., Lave, 1988; Mack, 1990). Mack (1990) refers to this type of knowledge as informal knowledge. She defines informal knowledge as "knowledge related to real-life situations that students construct and bring to instruction..." (p. 16) and contend that "both children and adults possess a rich store of informal knowledge" (p. 16). This informal knowledge may be either correct or incorrect and has a direct influence on our performance in a variety of mathematical content areas.

Gelman's (1980) contention that there are certain universal number concepts that occur naturally in normal people reflects the notion of informal knowledge. Gelman (1980) suggests that "young children know some things about number without the benefit of school instruction" (p. 54) and the concept of counting may be one of them.

Fischbein et al. (1985) reflect the idea of informal knowledge in their discussion of primitive models. According to Fischbein et al. (1985) primitive models are unconscious, intuitive models that dominate each of the fundamental operations. These models are stored in our unconscious and surface when we attempt to perform one of the fundamental operations. For example, the primitive model that most people possess in regard to multiplication is that of repeated addition.

Informal knowledge has also been noted in the studies of partitioning by Pothier and Sawada (1983). They argue that skill in partitioning requires a gradual progression through five levels - sharing, algorithmic halving, evenness, oddness, and composition. The sharing level "is learned by a child in a social setting" (p. 311) which corresponds to Mack's (1990) theory of informal knowledge. The other levels of partitioning in the five-level theory develop from the sharing level. This progression through levels in order to become skillful at partitioning is consistent with Mack's (1990) contention that children can build on informal knowledge to give meaning to formal procedures.

Mack (1990) has indicated concern about a conflict between informal knowledge and rote procedures. She suggests that the natural or informal knowledge the child has developed may be pre-empted by pressure to conform to a teacher-taught algorithm. She proposes that knowledge of rote procedures often interferes with informal knowledge. This interference is illustrated when a child tries to solve a problem by remembering a memorized algorithm and disregards his/her initial intuition regarding a solution. This is often the case with rational numbers. In traditional classrooms, students use teacher-taught algorithms and may not stop to think if their answer seems reasonable in the sense of being connected to their informal knowledge.

Lamon (1992) has addressed the intuitive aspect of the concept of unit. In her work with sixth grade children, she examined the thinking processes involved in solving ratio and proportion problems before receiving any formal instruction. These processes are explored through determining the ways in which children form units and determine a common unit. Lamon (1992) refers to the formation of composite units as "unitizing" (p.5). Unitizing is "the ability to construct a reference unit or a unit whole" and the ability "to reinterpret a situation in terms of that unit" (Lamon, 1992, p.6) is called norming. According to Lamon (1992) unitizing and norming "appear critical to the development of increasingly sophisticated mathematical ideas" (p. 6).

To illustrate Lamon's (1992) process of unitizing and norming consider one-third plus three-fourths. The addends can be unitized as 1(1/3-unit) + 3(1/4-unit)s.

Norming takes place when the units are reinterpreted in terms of a common unit. For example, one-third can be reinterpreted as 4(1/12-unit)s and one-fourth can be reinterpreted as 3(1/12-unit)s. The problem now becomes 4(1/12-unit)s + 3(3(1/12-unit)s-unit)s which is reunitized as 4(1/12-unit)s + 9(1/12-unit)s. Four of one unit plus nine of the same unit gives thirteen of these units, or in this case 13(1/12-unit)s or 13/12. This is similar to the way a child might join three groups of four apples and one group of six apples. The three groups of four apples can be reinterpreted as six groups of two apples and the one group of six apples can become three groups of two apples. The child has transformed the original groups of apples to nine groups of two apples.

Lamon (1992) contends that unitizing and norming play an important role in the concept of rational number - "In rational numbers, we not only create new unit wholes by composing units again and again but we also norm against those unit wholes" (p. 10). She suggests that analyzing ratio and proportion through the framework of unitizing and norming will provide insights into "the critical relationships we would like children to understand" (p. 12).

Lamon (1992) conducted a study with twenty-four sixth graders who had no prior formal instruction in ratio and proportion. Analysis of the study focussed on five problems which were designed to explore children's ability in unitizing and norming. Personal interviews were used to evaluate student performance and the audio-tapes of these interviews were used to evaluate the use of unitizing and norming.

Results of the study supported the notion that children possess an informal understanding of ratio and proportion. The successful strategies of the students demonstrated a strong intuitive application of unitizing and norming - "Three-fourths of the students interviewed naturally formed ratios of otherwise unrelated sets and engaged in the process of norming, or reinterpreting one ratio in terms of the other" (p. 31).

The above discussions have confirmed the existence of informal or intuitive knowledge (Lamon, 1992; Mack, 1990; Fischbein et al. 1985). Work by Pothier and Sawada (1983) and Mack (1990) has indicated that informal knowledge can be used as a foundation for further knowledge construction and meaningful procedures. The informal knowledge of unit formation can provide the foundation for extending the learners' knowledge of the whole number domain to that of the rational numbers.

Von Glasersfeld (1981) suggests that the formation of units is a natural instinct that occurs in various aspects of our daily lives. This intuitive notion of unit provides the operational basis for the construction of number. Students begin counting by ones but later extend to counting by twos, fives, tens, etc. as the unit concept develops. Whole numbers become conceptualized as the composition of various units (e.g., the number 6 as 1 + 5, 2 + 4, 1 + 2 + 3, etc.). This initial focus on the unit can extend to the conceptualization of rationals as compositions of various units (e.g., 3/4 as 1/2 + 1/4, 1/4 + 2/4, 1/4 + 1/4 + 1/4, etc.) thereby laying a foundation for meaningful computation algorithms.

Conclusions

The research presented in this chapter has been broad. The first part of the chapter focused on describing the rational number dilemma faced by elementary and middle grade students and their teachers. Then discussion shifted to the role of the unit fraction in this dilemma. The unit fraction as discussed here is merely an extension of the more informal notion of unit. Finally this chapter confirmed the existence of informal knowledge and the notion of building on this knowledge to attain conceptual understanding. The research presented supports the current study in the following ways:

(a) Students' awareness of their informal knowledge regarding the unit concept promotes understanding; (b) teachers who provide opportunities for students to build on their informal knowledge by working with various whole number units to develop unitizing and norming skills help students develop schemes for further work with rational numbers; and (c) students who become accustomed to focusing on the unit may more readily recognize intuitive and authentic connections between natural and rational numbers.

CHAPTER THREE

METHODOLOGY

The intent of this study was to follow and describe the cognitive processes of five prospective elementary teachers as they engaged in the formation of units and to examine the role of the unit concept as a possible link between whole number and rational number concepts and operations. I was trying to gain an understanding of how the subjects' constructed units and whether or not their attention to and understanding of the unit concept would increase their understanding of rational number concepts and operations.

Research Questions

This study investigated the conceptualization of the unit as a way to bridge the gap between whole numbers and rational numbers. It was my contention that by building on the informal knowledge of the unit concept a natural connection could be established between whole numbers and rational numbers, thereby aiding in the conceptual understanding and reducing common misconceptions associated with rational numbers. I did not have specific testable hypothesis, but the following questions were considered:

1. Do preservice elementary teachers exhibit informal knowledge regarding unit formation?

2. Are they aware of their own construction of units?

3. What cognitive obstacles are encountered in the process of understanding the unit concept?

4. How will an awareness of the intuitive nature of the unit concept affect their problem solving performance on whole number addition and subtraction?

5. Will knowledge of the role of the unit concept in the whole number domain facilitate learning of concepts in the domain of rational numbers?

Background

My interest in this research stems from my previous work with prospective elementary teachers. For the past ten years I have taught the mathematics content course for elementary education majors. During this time I have been forced to recognize just how little conceptual understanding these future teachers possess in regard to rational numbers. As we manipulate fraction bars, fold paper, and partition area models in an attempt to gain understanding, students are still unable to demonstrate conceptual understanding when asked a "why?" or "How do you explain this?" type of question. They have memorized the algorithms of rational numbers but have no clue about the concepts. Their rational number knowledge has been acquired in isolation rather than as an extension of the more conceptually understood domain of natural numbers. The word fraction seems to be synonymous with alien, as being something foreign or unfamiliar, therefore rational numbers are considered outside of rather than derived from or as an extension of the set of natural numbers. One of my primary goals in this research was to examine the possibility of the unit concept as a missing link between the natural and rational number domains.

Since the focus of this study was on the cognitive processes of the student, a qualitative methodology was appropriate. A type of qualitative approach now used frequently in mathematics education research is the teaching experiment. According to Vygotsky (1962) the teaching experiment was designed for "the student of concept formation" (p. 52). Since my objective was to follow the students' construction of the unit concept in a learning environment, the teaching experiment was chosen as the research method for this study.

Teaching Experiment

Before looking at the specifics of the research components I feel that it may be appropriate to justify the selection of the teaching experiment as the method of research. This method was first proposed by Vygotsky in the 1920's and has been used successfully by many prominent researchers: Steffe and Spangler (1993); Simon and Blume (1992); Behr et al., (1984). The dynamic nature of the teaching experiment allows one to observe intellectual development while determining how instruction can best influence this development. The method combines interview and observation with a flexible teaching component. The teaching component consists of a sequence of lessons which are structured before the experiment but are modified continually as unanticipated problems or new insights arise during the experiment. Since attention to the concept of unit as presented in this study was new to the subjects and the researcher the flexibility provided by the teaching experiment was essential.

In the teaching experiment the researcher is both the teacher and the observer. The researcher structures and modifies the teaching component based on personal observations. Since the researcher is also the teacher, avenues of interest that may appear during a lesson may be immediately explored.

I, the researcher of the study, was also the teacher for the teaching experiment. However, I was not the teacher of the original class from which the students were selected. This arrangement was followed in an effort to control the conflict which could occur when the researcher is also the one responsible for the grade. My intention was to make the students feel more comfortable expressing negative responses or reactions since they knew they were not being evaluated for a grade.

Sample

The sample for this study was selected from a course in mathematics for elementary education majors. This course provided an appropriate population since it is the first mathematics course encountered by the elementary education major that is designed solely for the prospective elementary teacher. The students enrolled in this course usually have limited backgrounds in mathematics and generally exhibit a dependence on memorized algorithms rather than conceptual understanding.

A group of five students were selected from a class enrollment of approximately forty-five. Due to the nature of the study an intentional selection was used rather than a random selection. An intentional selection process was chosen to insure that the initial pool of nine students was as much like the regular class as possible. The features considered in selecting the nine students included age, male/female ratio, and mathematical ability.

The selection process consisted of an inventory (see Appendix A) and individual interviews. The inventory items were designed to differentiate among students in their construction of units and rational number concepts.

The inventory consisted of ten questions and was administered to the entire class. Each question was on a separate sheet of paper. The students were first given questions one through four and asked to find the solutions and show their work. These four questions consisted of an addition of fractions with unlike denominators, a word problem that necessitated units of one, a word problem that could be solved by various units, and a word problem with fractions that could be solved by various units.

After completion of the four questions the students were given six more questions (each on a separate sheet). These questions were similar in content to the original four questions. The students were asked to sort all ten questions into categories. Working the last six questions was optional. They were provided with a sorting sheet on which to record the groups and explain why they grouped particular problems together. This sorting activity allowed the researcher to determine if any student grouped the problems based on the possible unit formations. This was not the case. Grouping was based on superficial clues like "contains a fraction" and "does not contain fractions."

The inventory was analyzed and nine students were identified as being representative of the various levels of unit construction and rational number understanding revealed by the inventory. These nine students were also selected to best simulate an ordinary classroom and therefore consisted of students with different aptitudes for mathematics (above average, average, and below average) and different classifications (i.e., sophomore, junior, senior). This information was provided by the classroom teacher. The nine students identified consisted of eight females and one male. This was not surprising since this class is almost always entirely female.

Letters of inquiry were sent to the nine students to describe the details of the study and to determine their willingness to participate. Only five of the female students responded but these five students were representative of the entire class. Individual interviews were conducted with the prospective students. The purpose of these interviews was to determine the communication capabilities of the student and their willingness to express and attempt to explain their thinking. During the interviews it was noted that some of the students were more verbally capable than others. One student in particular seemed very reluctant to verbalize her thoughts. After analyzing the interviews it was decided that all five students would become the focus of the study. The five students will be referred to as Ann, Carolyn, Mary Gail, Lisa, and Renee.

Ann was a twenty-two year old sophomore interested in teaching in grades one through four. She was mathematically the strongest student participating in the study. Her success with mathematics is revealed in her mathematics autobiography. Ann writes, "I always had positive experiences in math. Math came very easy for me. I always made A's and caught on really easy." Ann's background included algebra, geometry, trigonometry and analytical geometry. She indicated that she always wanted to know "exactly why problems are worked a certain way...Even if I can work the problem perfectly, I get really frustrated if I don't understand what it means."

Carolyn was a twenty-four year old junior who was deciding between two majors, elementary education and social work. In high school she had two years of algebra (passed one), one year of geometry, and one year of business math. She indicated that she was generally an average student but never had "a full concept of algebra." The two developmental mathematics courses she took in college helped her develop the fundamental of algebra, she writes, "For me math did not "click" until

Math 91. Taking Math 91 and Math 92 did me a world of good."

Mary Gail was a thirty-four year old senior interested in teaching in grades one through four. She was married and had one child, with another on the way. In high school she took one geometry course and one algebra course. Mary Gail indicated that she "scraped through" her math courses with average grades but attributed this to lack of interest. In describing her elementary school math experience she was taught in "a very methodical and organized fashion." She indicated a lot of memorizing and use of flash cards. Her most positive math experience to date was in the mathematics course for elementary education majors in which she was currently enrolled. She writes, "Each new concept I learn; I finally see a little light bulb over my head and hear an "A-HA!" inside my head!"

Lisa was a twenty-four year old senior interested in teaching in grades one through four. In high school she had taken two years of algebra, geometry, and trigonometry/advanced math. While her list of courses sounds impressive, Lisa made a confession in her essay, she writes, "I never really understood what I was doing or why I was doing it." Lisa considered herself to be an average or above average student. In her first semester of college all of her previous math courses seemed to pay off, she writes, "I still don't know what my 161 teacher said or did but he definitely turned the light switch on."

Renee was a twenty-three year old senior interested in teaching in grades one through four. She gave no indication of her high school math experiences but she had taken two college developmental math classes which suggests a limited background. Regarding her mathematical ability, Renee writes, I've never been able to grasp what I was supposed to do." In her autobiography Renee focused on her elementary school experiences. She indicated a dislike of mathematics that originated from fear. Renee attended a very small elementary school and had the same teacher for grades three, four, and five. Renee's dislike of math can be traced to this teacher, she writes, "She instilled such a fear in me that it made it difficult to concentrate on the lessons." Renee indicated that she did learn her basic facts, "but it was only out of fear of what she may do if I did not learn them."

Teaching Sequence

The teaching sequence was conducted over a three week period and consisted of five 50-minute group sessions and an individual teaching interview. The period of three weeks, while relatively short, was established to mirror the amount of time spent on rational numbers in the regular class from which the students were chosen. The group of five students met with the researcher twice a week instead of attending their regular class. This schedule was followed to insure that none of the students would receive additional instruction on rational numbers or the unit concept for the duration of the experiment and that all exposure to unitizing and rational numbers was observable. The sessions were held in a private room equipped with a video camera.

Each session consisted of three parts - instruction (presented in the form of a teaching-interview), observation, and discussion. The instruction segment was used to summarize previous lessons and to introduce the planned activities for that day. The teaching-interview format was used since this segment was not meant to be a lecture but rather an interaction between the students and the researcher.

The observation segment allowed the researcher to view the students' involvement and reaction to the activities. The researcher observed the students and took notes on the students' comments and procedures.

Further observation was made possible via the video camera.

Most of the activities were of an individual nature so the researcher could observe the individual reactions. Interactions among the five students took place during the discussion segment. The discussion segment allowed the students and/or researcher to verbalize their reactions and questions concerning the activities or their observations.

While the five lessons were pre-structured, the observation of each daily lesson was used to modify the pre-structured lesson for the next day's session. The videotapes were reviewed at the end of each day as well as the observation notes to determine if any changes were necessary for the next day's session (see Appendix B for the actual lessons).

Data Collection

Due to the qualitative nature of this study data were gathered from a variety of sources: videorecordings of group and individual sessions; audiotapes of individual interviews; written responses to instructional materials; student essays; and the set of journals kept by the researcher throughout the entire teaching experiment. The various sources of data are presented in Figure 1.

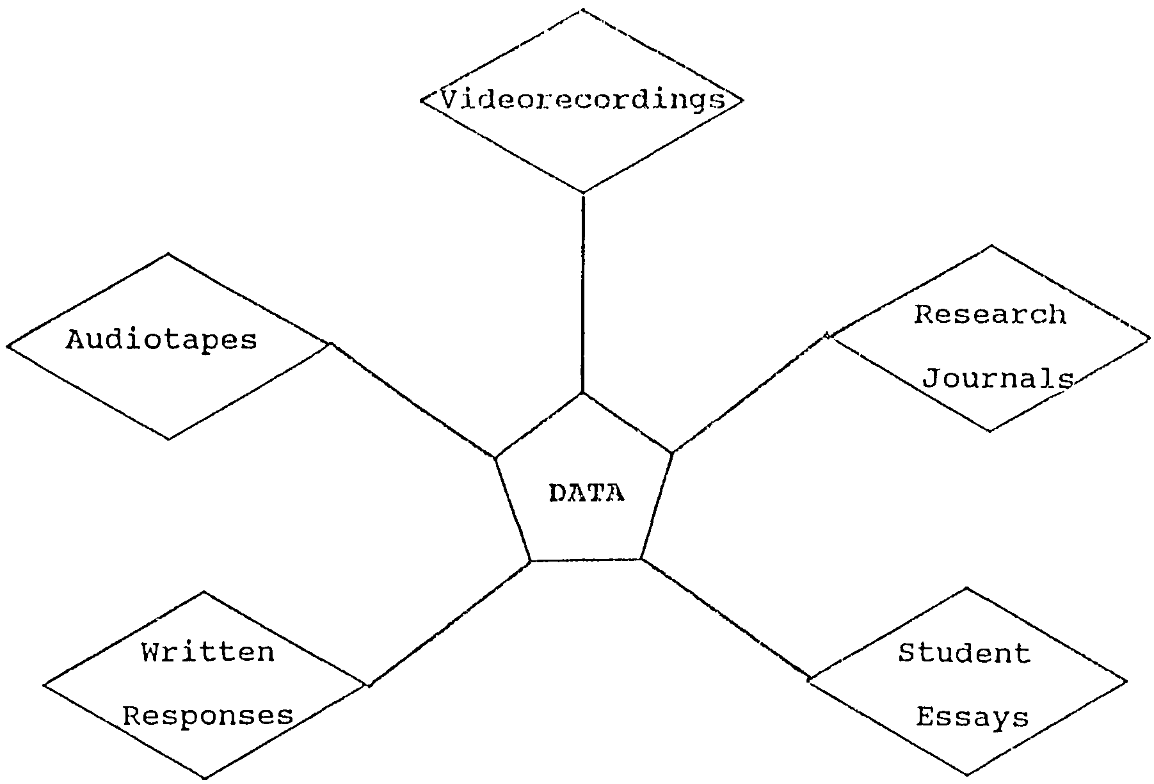


Figure 1. Data collection

Videorecordings.

Videorecordings were made of the individual teaching interviews as well as each of the five group sessions. These tapes were analyzed daily and cumulatively. Any modifications for the next day's lesson were based on these analyses. By providing a visual as well as an audible account of the daily activities, the researcher had a second chance to view reactions or developments that were missed during the daily session. The videorecordings were made by a non-participating student, and the researcher was visible throughout the tapes so that all interactions involving the students and/or the researcher were observable.

Research journals.

The researcher kept two journals throughout the experiment. One of the journals was used to record observations made during the daily lessons and while viewing the videotapes. It was used to develop future tasks, modify lessons, and record comments or actions of the students. The other journal consisted of reflections. The writings for this journal also occurred daily and cumulatively. The writings consisted of emerging patterns, insights, questions, problems, ideas, etc. encountered by the researcher.

Audiotapes.

Audiotapes were made of the interviews during the selection process. Although notes were taken during the interviews, the tapes allowed the researcher more freedom in reflecting on the comments made by the students.

Student essays.

Students were asked to write two essays during the three week period. The first was a mathematics autobiography. In this essay students were asked to write about their feelings and attitudes concerning their encounters in previous mathematics courses. For the second essay students were asked to write about a time in mathematics when all of a sudden something clicked. These essays provided the researcher with information concerning the attitudes and beliefs of the students. Since the researcher had no prior contact with any of the students participating in the study, any familiarity with the students' previous experiences with mathematics would increase the researcher's sensitivity to variations in behavior or attitude that might occur during the teaching experiment.

Written work.

The written work consisted of the preinstruction inventory, daily activities and homework assignments. This provided the researcher with a tangible source of student performance and progress.

Data Analysis

The analysis of qualitative data is never an easy task. To aid in the process several sources were consulted. The guidance of Bogdan and Biklen (1982), LeCompte and Preissle (1984), the readings of other qualitative studies, and conversations with experienced qualitative researchers provided the foundation for my analysis.

My analysis can best be described as a three stage process. The first stage took place during the actual study. The daily reflections and writings kept me mindful of my research goals and provided gradual insights for future analysis. The daily readings of the students' work helped me become sensitized to the behavioral and conceptual changes of the students and provided clues for continued focus.

The second stage in the analysis of the data involved the complete transcriptions of the individual and group lessons. I found this to be a long and tedious process. After completing the transcriptions most of my time was devoted to continuous reading of the transcripts in order to familiarize myself with the data. Gradually patterns and themes began to emerge.

The third stage was an attempt to organize the data into categories by using a coding system similar to that suggested by Bogdan and Biklen (1982). The initial coding system was based on the research questions and then a recoding of the data emphasized emerging themes. Although a bit primitive, the coding systems involved color coding each research question or theme and then perusing the transcripts to highlight supporting data in the corresponding color. When the writing phase began, supporting material for the research questions and themes was easily determined. A lesson by lesson analysis was also conducted to reveal individual progressions.

CHAPTER FOUR

RESULTS OF THE LESSONS

In order to show the progression of the individual students in the conceptualization of the unit concept this chapter will describe the content of each lesson and provide an analysis of the individual responses.

The Lessons

In the five lessons that follow the students were exposed to many facets of the concept of unit. While there is no universal definition of unit established in the literature, throughout this study the word unit will be used to describe a grouping that is conceptualized as a whole or entity. Indications of this conceptualization will be determined through verbal communication, by the markings on the students' written work, and/or by physical gestures such as hand motions observed on the videorecordings.

Throughout the lessons the students were asked to manipulate unit quantities from different perspectives. In some of the lessons the number of units was fixed and the student controlled the size of the unit. In other situations presented in the lessons the size of the unit was fixed and the student controlled the number of units. In lesson 4 the student controlled both the size and the number of units.

As discussed in chapter 2, many researchers contend that the conceptualization of a fraction as a unit evolves from the act of physical or mental measuring (e.g., Saenz-Ludlow, 1994; Olive, 1993; Hunting, 1983). Lesson 5 of the teaching experiment was designed to provide such experience.

Other aspects of the unit concept that were addressed in these lessons included unitizing, reunitizing, and norming. In order to insure that the reader has a clear understanding of what is meant by each of these terms, a few lines will be devoted to their description. Unitizing refers to a conceptual process of splitting or uniting one or more groups in order to form one or more units. Once units are formed, reunitizing occurs when the original units are restructured to form new units or composite units. Norming refers to the process of imposing a common unit structure on one or more units. For example, suppose 12 cubes are placed on a table. The initial focus on the cubes can be described as 000000000000. Suppose the child decides to count the cubes by twos. There is a refocus that can be described as (00)(00)(00)(00)(00)(00). This refocus illustrates the process of unitizing. The selection of two as the means of counting illustrates the idea of norming. Suppose the child groups the cubes as (00 00 00) and (00

00 00) this would indicate that the child had reunitized the original six units of two to form two units of six. The aspects of unitizing, reunitizing and norming will appear throughout the discussions of the lessons.

Lesson One

The first lesson of the teaching experiment was designed to reveal a) whether or not the students would spontaneously group or unitize objects to be counted and b) to establish recognition of how broadly the concept of unit pervades everyday activity. It should be noted that no mention of grouping or units was made before the lesson. The students were merely asked to count. This lesson involved four counting tasks. These tasks ranged from actually counting a bucket of cubes, where students could touch the objects, to estimating the number of heads in a crowd simulated by dots on a poster that was displayed out of reach to the students. The purpose of this activity was to examine if and to what extent the students formed units other than one.

Counting cubes.

In the first task of lesson 1 each student was given a bucket of cubes and asked to count the cubes and record their answer and their counting procedure on the task sheet provided. Each bucket contained at least 200 cubes which included individual cubes as well as various snapped stacks of cubes. The stacks ranged in size from two cubes to about six cubes.

Mary Gail used a variety of units in her counting procedure. She began by taking all of the stacks from the bucket, counted each stack, and then added the counts of the stacks together. This yielded an even number so she began to count the individual cubes by twos. Whenever she reached 100 she would push those cubes aside, focus on the remaining cubes and begin counting starting with one. She ended with two piles of 100 cubes and one pile of 38 cubes.

In analyzing Mary Gail's procedure one notices units of various sizes; units corresponding to the stacks, units of 2, units of 100, and a unit of 38. Mary Gail did not un-snap the stacks, but instead treated each stack as a unit or whole. When she first began to count by twos she would snap two cubes together to form a stack, but she soon tired of this extra effort and began to grab two at a time. These original units became embedded in the larger unit of 100 as Mary Gail made her piles. In describing her procedure Mary Gail writes "I started out by adding cubes that were stuck together, then I added by groups of 2. (I made 2 piles of 100 and one of 38). I took the cubes out of the bucket as I counted." Mary Gail's final procedure can be described as 2(100 cube unit)s + 1(38 cube unit).

Lisa, Renee, and Carolyn used procedures that were similar to each other. Each one normed the bucket of cubes by creating stacks of five. This was done by snapping five single cubes together or by adjusting an existing stack. For example if the original stack had six cubes, one of the cubes would be removed to produce the five-unit, whereas if the original stack had less than five cubes then additional cubes were added to produce the five-unit. Once the cubes were normed Lisa grouped by twos and counted as they were returned to the bucket, she writes "I counted the columns in tens instead of fives (such as 10, 20, 30...)." Her use of the word column provides further evidence that she conceptualized the five cubes as a whole or a unit. Her counting by tens indicates the two 5-cube units were reunitized to form a 10-cube unit. Lisa's total of 224 cubes can be described as 22(2(5 cube unit)s-unit)s + 4(1 cube unit)s or 22(10 cube unit)s + 4(1 cube unit)s.

While Carolyn used the same five-unit procedure as Lisa in grouping her cubes, she used a different approach to counting as she returned the cubes to the bucket. Carolyn writes, "I put the cubes in stacks of fives. I then counted the stacks of cubes by 5, 10, 15, 20...as I put them back into the bucket. I had one cube left over, that would not go into a group of five. I added that cube to my total." Carolyn's 216 cubes were conceptualized as 43(5 cube unit)s + 1(1 cube unit).

Renee obtained her total of 207 cubes through five-units but her approach was different than either Carolyn's or Lisa's.

Renee: I did the same thing they did with the fives. I just grouped them in fives and then just threw them in fives in the bucket.

Tena: When you put them back in the bucket, did you group them by tens?

Renee: No. I left them in fives...I just counted the number of fives I had then multiplied. The total I had was 41 groups of five and then I multiplied it by 5.

Renee's indication that she "counted the number of fives" supports the contention that she was conceptualizing 5(1-unit)s as 1(5-unit). Further unitizing occurred as the 41(5 cube unit)s became a single unit after her multiplication: 1(41(5 cube unit)s-unit) or 1(205 cube unit). Her total of 207 cubes can be described as 1(1(205 cube unit) + 2(1 cube unit)s-unit).

Ann is the only student that used units of one:

Ann: I just dumped them all out and started counting by one, and if I got to a group of five, I mean a group of like less than, if it was two or three or maybe four, I would add that to it, but if it was probably five, I'd just say I'm on 16, I'd just go 17, 18, 19, 20, 21. But towards the end, I started when I had just individuals left and I was on an even number, and I would just count by twos. It was faster.

Tena: Kind of a mixed strategy?

Ann: Yes. But I really just counted by ones.

While Ann insists that she "just counted by ones," other unit structures were involved. If she came to a stack with less than five cubes she treated this as a whole or unit and increased her count by that number. This suggests that Ann also used units of two, three, and four. Toward the end of her counting Ann's focus turned to units of two. Her use of various units seemed to cause some doubt, Ann writes, "I kept worring [sic] that I might have miss counted and wanted to start over or re-count to make sure I was correct, but I didn't."

While it is impossible to tell Ann's exact procedure one description of Ann's 204 cubes might be
100(1 cube unit)s + 2(3 cube unit)s + 1(4 cube unit) + 47(2 cube unit)s.

<u>Counting faces</u>.

On the second task of lesson 1 the students were asked to count a group of 30 faces (see Appendix B) that were printed on a sheet. Ten faces were scattered on each of three horizontal rows. The students were asked to determine the number of faces and then describe the process they used to count the faces.

All of the students counted the faces by using units of one except for Renee. In describing her method Renee writes, "I grouped the faces into groups of 5, then I counted the number of groups (6) and multiplied them together and got 30." Renee's focus on units of five is clearly visible on her task sheet where each group of five faces has been circled. In discussing her strategy with the other students Renee declared, "I grouped. I group everything...It's easier for me."

The students were then asked to count the faces again using a different approach. Lisa's second approach still focused on units of one only this time she counted diagonally rather than left-to-right. Mary Gail describes her second attempt, "I grouped in three groups of 10...That wasn't really too different than what I did, though, the first time."

Counting sticks.

The third task involved counting a bag of popsicle sticks in which the sticks were grouped in various sizes by the use of rubber bands.

Renee was the only student that completely unbanded all of the groups. In describing her process Renee writes, "I counted the sticks by grouping them into groups of 10's. I had 5 groups of 10 with 9 singles remaining (i.e., 5(10-unit)s + 9(1-unit)s). I multiplied the groups of 10 by 5, and got 50 (i.e., 1(50-unit)). To this 50, I added the 9 singles. This gave me a total of 59." Renee's method began with the reunitizing of each 1(bundle unit). She treated the original sticks as single units as indicated by her approach of unbanding the groups, counting to 10, and then re-banding to form groups of 10. These new bundles of 10 were then conceptualized as (10-unit)s. Her total collection of sticks was found by 5(10 stick unit)s + 9(1 stick unit)s which became 50(1 stick unit)s + 9(1 stick unit)s = 59(1 stick unit)s.

Lisa's approach to counting the popsicle sticks was based on units of one. Although she did not unband the groups she still treated the sticks as single units. She seemed to look inside of each bundle and reunitize the 1(bundle unit) to x(1-unit)s. Lisa writes, "I counted the popsicle sticks one-by-one and kept them in the bundles. I also counted them twice to make sure I had counted correctly the first time. I felt as though there was no need to take them out of the bundles." Her process can be described as forming 59(1 stick unit)s.

Mary Gail, Ann, and Carolyn used similar approaches. Each one essentially left the sticks in the bundles, counted the number of sticks per bundle, and then added these together. While their final procedure may be described as 1(3 stick unit) + 1(8 stick unit) + 1(7 stick unit) + 1(14 stick unit) + 1(9 stick unit) + 1(6 stick unit) + 1(12 stick unit), this does not indicate the complexity of the unit structure they used. The focus on a group of three sticks as 1(3 stick unit) can be explained by the notion of subitizing. However to view eight sticks as a unit (i.e., 1(8-unit)) involved both unitizing and reunitizing. The eight sticks were first counted one at a time to indicate the notion of 8(1 stick unit)s. These eight single units were then reunitized as 1(8 stick unit) to be united with the other units. This thought process was also used for the other units whose size was greater than three.

Again the students were asked to consider a different approach as indicated by the following dialogue:

Tena: If I asked you to do it a different way, after hearing everybody's strategies, do you think there would be an easier way that you could do it this time?

Lisa: I would unbundle and group.

Mary Gail: Yes, that's probably what I would do.

Ann: Kind of like she did [refers to Lisa's original approach of continuous counting], just keep going.

Carolyn: I would do groups because it's easier to go back and recheck yourself.

Most of the students agreed on an approach like Renee's in which a norming process was used. The discussion continues as the choice for a norm is considered:

Tena: What do you think you would group by?

Carolyn: Tens.

Tena: Would you change your grouping? Like the more sticks I put on the table would you change to a higher group or a smaller group if I took away some sticks, or do you think you would just stick with tens?

Ann: I'd do tens until I got to 100 and then I'd set it aside.

Mary Gail: Yes, and then set that pile aside so you'd have groups of groups.

Carolyn: Like 10 groups of 10.

Renee: Start small and then make a bigger pile out of that.

Tena: So kind of adjust your unit as you go up as far as tens and then put your tens together to make hundreds and then put those together? Okay.

The above discussion illustrates that the students were conceptualizing composite units. The phrases such as "groups of groups" (Mary Gail), "10 groups of 10" (Carolyn), and "start small and then make a bigger pile" (Renee), suggest that the students were beginning to recognize and use the nesting property that occurs in composite units.

Counting a crowd.

In the final task of lesson 1 the students were asked to estimate the number in a crowd simulated by random dots on a poster. The task read as follows:

> Please refer to the poster on the table. Pretend that you are standing on top of a tall building and you are looking down at the crowd below. The dots on the poster correspond to the heads in the crowd. You need a good estimate of the number of people for the newspaper. How many people are in the crowd?

Although the task sheet indicated the poster was on the table, the poster was actually displayed on the wall. This change was made so that the students would not be tempted to mark on the poster or physically touch the poster.

All of the students used units other than one, which is not surprising considering the nature of the task.

The approaches seemed to combine a geometric-doubling process. This approach can best be seen in Mary Gail's strategy.

> Well, I saw hearts, sort of in the shape of a heart or circle, and I took the very top left corner and counted out 20 dots. And then I figured there were about...I saw approximately 11 circles or hearts on the top half; so I multiplied 11 times 20, and then I just said, well, it looks kind of evenly distributed so I just doubled that figure and I came out with 440.

Mary Gail essentially saw a geometric figure, a heart or circle, counted the dots in that figure then visually repeated the figure across half of the poster and finally doubled that amount. This strategy supports von Glasersfeld's (1981) thesis about the formation of units by "segmenting" from the background. The nesting of units used in Mary Gail's approach could be described as 20(1 dot unit)s which was conceptualized as 1(20 dot unit) which was repeated and became 11(20 dot unit)s and then 1(11(20 dot unit)s-unit) and finally 2(11(20 dot unit)s-unit)s.

The geometric figure used by Ann, Carolyn, Renee, and Lisa was a rectangle. Lisa visually formed two rectangles by dividing the poster in half vertically. Within one of the rectangles she used a spiral technique to count the dots in that half and then "multiplied that number by 2 to get 220" (Lisa).

Ann also divided the poster into vertical halves. Then she identified a small rectangle at the top left half, Ann writes, "I counted 30 dots in that group. Then I counted 5 even lengths down the poster. Then multiplied by 2 because I was only using have [sic] the width. That gave me 10, so I multiplied 30 dots time 10 groups."

<u>Discussion</u>.

Discussion among the students took place at the completion of all four tasks. This format was followed so as not to suggest the notion of unitizing to those students who worked with units of one. As the various units used by the students in the counting tasks became apparent through the discussion, the students were asked to think of another situation in mathematics in which units were formed.

Mary Gail: Do you mean like when we got into multiplication and division and all that?

Tena: Yes. Did you think about grouping or were they more just memorized?

Renee: With multiplication I think it was grouping, because it was easier...

Mary Gail: I think subconsciously we were grouping. If it was four times nine, we'd want four groups of nine.

Tena: Like repeated addition?

Mary Gail: Right. But when I went to school it was strictly memorization, but I think subconsciously we were all doing it to help us remember.

It is interesting to note at this point Mary Gail's use of the word subconscious. This suggests a natural or intuitive use of unit structures; knowledge that Mack (1990) calls informal knowledge.

The students were then asked to think of another discipline or other situations aside from mathematics, where the formation of groups creates units.

Renee: When you count money.

Tena: Okay. When you count money. How would you count like a roll of pennies or a roll of nickels? How would you do that?

Renee: I've always just recognized, like if I counted pennies, because you need 100 say, just break them in groups of 10 and put them in stacks.

Carolyn: I do mine in fives.

[Discussion of counting money continues until students were asked to think of another situation.]

Mary Gail: In elementary school?

Tena: You can think of anything. What do your closets look like?

Carolyn: An explosion of clothing.

Lisa: Shirts, dresses. All the pants are here. The shirts are here in groups.

Mary Gail: Food cupboards or pots and pans. I have like my food cupboards, I have all my baby food on the lowest shelf. You know and I, I'm pretty organized. Fruit, vegetables, meat.

Renee: I think you group at the grocery store whenever you're putting your groceries when you're getting ready to check out. I like to put my box stuff here, and I take the cold stuff and I'll put it here and cans. I group there.

Lisa: I group my books in my book bag.

The students' ability to think of non-mathematical situations in which unitizing occurs demonstrates a broadening of their recognition of the role of the unit concept. There was a sense of excitement as the students realized more and more situations in which units were formed. Entries from that night's journals revealed the presence of unit structures in the following situations: organizing money in your wallet; separating your clothes, like a drawer for socks and a drawer for shorts; a six-pack of cokes; and organizing medicine in the medicine cabinet. One of the nicest examples was from Renee, she wrote, "When you order large quanities [sic] of pictures they are often grouped into units. Such as, 1 8x10, 2 5x7s, and 8 walletts [sic] mean 1 unit."

The students' procedures and comments regarding lesson 1 supported the notion that the tendency to unitize is indeed intuitive. Through the discussion, the students became aware of this natural instinct by disclosing examples of various units used in mathematics as well as in other areas. The focus of the research then became to what extent would the students use this intuitive knowledge in the remaining lessons.

Lesson Two

The second lesson of the teaching experiment was designed to indicate if the awareness of unit structures obtained in lesson 1 would affect the solution process of problems involving whole number operations. Each of the three tasks in lesson 2 contained a mathematical word problem. The first task presented a problem along with a diagram that could be used to simulate the problem. The second problem was presented along with physical objects that could be used to act-out the problem. The purpose of the diagram and the physical objects in the first two problems was to provide a realistic setting in which the problems could be solved, much like the tasks in lesson 1. The last word problem was given as it might appear on a test or in a textbook with no diagram or physical objects. The purpose of this lesson was to compare and contrast the unit formation of this lesson with that used in lesson 1.

<u>The donut problem</u>.

The following problem was given to the students:

> A local bakery has developed a new plan to improve the sale of donuts. Every morning donuts were boxed by the dozen in preparation for the morning crowd. By mid-day many of the boxes remain unsold. In an effort to promote the sale of donuts after 11:00 am, the manager has decided to sell donuts by the snack-pack. Any box left unsold after 11:00 am will be re-packaged as snack-packs and sold at a reduced price. If a snack-pack is to contain three donuts, how many snack-packs can be made from four boxes of unsold donuts?

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The unit structures used by the students in their solution process included units of 1, units of 12, and units of 3. Mary Gail used a unit of one approach as she converted the four dozen donuts to 48 donuts. Then she divided by 3 to get 16 snack-packs. She indicated that she did use the diagram but only to check her division. A physical analogue of Mary Gail's approach is that she essentially opens the boxes and dumps all the donuts in one pile to form one large unit of 48 donuts. Then she focuses on units of three as she divides to determine the number of snack-packs.

Lisa indicated that she used the diagram at first but only because she felt she was supposed to. She describes her approach in the following dialogue:

Lisa: Then I just divided 12 by 3 to get four snack packs per dozen and then times that by 4 dozen to get 16.

Tena: So you kind of did snack packs within a dozen and then did it?

Lisa: Right.

Ann and Lisa both focused on one dozen as a unit. Within this unit they re-focused on units of three to determine the number of snack-packs in that dozen which they found to be four. Then they multiplied this result by 4 since there were four of the dozen-units. Symbolically their process can be described as an initial focus on 1(12 donut unit) then within this unit they reunitized to get 1(4(3 donut unit)s-unit) and finally expanded their focus to 4(4(3 donut unit)s-unit) to get 1(4(4(3 donut unit)s-unit)s-unit), 1(16(3 donut unit)s-unit), or 16(3 donut unit)s.

Carolyn used an approach similar to Ann's and Lisa's but she never focused on the dozen as 12 donuts. She writes, "I group [sic] the first picture in to [sic] 4 groups of 3, and multiplied by the number of boxes (4)."

During the discussion Carolyn describes her process as follows:

Carolyn: On the first dozen down here, I grouped them and saw that that was four. It didn't click that was 12. So I saw that there was four snack packs in each box of donuts, and then I just multiplied that by number of boxes which was four and I have 16.

Renee was the only student who used the diagram to solve and the arithmetic to check. On her task sheet you can see her repeated circling of three donuts. The circles indicate the conceptualization of three donuts as a unit within the larger unit of a dozen. This conceptualization of a unit is further supported by her physical motions on the video tape. As Renee discussed her process she pointed to each of the circles as if to indicate a whole. When asked what unit she used, Renee cupped her hands around the first dozen as she said, "I took the first dozen and saw how many units of three were in 12." This cupping motion suggests that each dozen was seen as a unit and within the dozen each circle represented a snack-pack unit. In describing her process Renee writes, "12 donut singles grouped into donut units of 3 gives you 4 snack-pack units per dozen. 12 ÷ 3 = 4 x 4 = 16 [sic]."

After discussing their procedures, the students were probed to determine if they were aware of the unit structure that they used. The dialogue with Mary Gail does not reveal her focus on units of one.

Tena: If I go back and ask you what unit you used, tell me what unit did you use?

Mary Gail: Three.

Tena: The first thing you did was take the four dozen and multiply by 12?

Mary Gail: Yes. I multiplied four times a dozen, and it came out to 48. And then divided that into groups of three.

Tena: So you pretty much looked at 48, you looked at four 12-units and kind of switched that to 48?

Mary Gail: Right.

Tena: And then from the 48 you looked at...

Mary Gail: How many three units there were.

Descriptions given by the other students seemed to reveal a better understanding. Lisa describes the units she used as "a dozen and a snack pack." Ann states, "I did the 12. In a dozen there's four three-units times 4 dozen units." Renee indicated, "I took the first dozen and saw how many units of three were in 12." The following description by Carolyn suggests a little confusion with the distinction between the content or size of her unit and the number of units: "I just grouped into the 3(4 unit)s and then multiplied 3 times the four larger units." The reversal that occurred in Carolyn's description was not present on her task sheet.

The comments of the students presented above reveal their attempts to use unit terminology. This provides further support for the contention that the students were becoming more conscious of the unit concept.

The bubble gum problem.

The second problem given in Lesson 2 included actual packs of gum so that the students could use a hands-on approach similar to those in lesson 1. The problem is given below.

> Susan is team mother for her son's baseball team. One of her duties is to provide bubble gum for the players. She has decided to buy the sugarfree gum that comes in regular packs five sticks and family packs of 18 sticks. Susan allows two pieces of gum for each player per game and buys just enough gum with no extra pieces. If she buys two five-stick packs and two 18-stick packs for the first game, how many players are on the team?

It is interesting to note that none of the students used the physical objects in their solution and all of the students solved this problem using the same approach. Essentially all students converted to units of one. Mary Gail's description is the most detailed,

Mary Gail: I just took two packs of five, multiplied that through and two packs of 18, multiplied that through, then added them together. I added my results together and came up with 46 individual pieces of gum. I think I see a pattern here. And then I divided it by how many each child would get, which is two, and I came out with 23 baseball players.

Mary Gail's remark, "I think I see a pattern here" indicates that she is beginning to notice her dependence on units of one. The dialogue continues,

Tena: So what unit did you use?

Mary Gail: Originally, I used five stick units and 18 stick units. And then when I got that answer, I divided it by 2 game player units to come up with 23 players.

Tena: But just like with donuts, you sort of essentially opened up all those packs and dumped them and then started picking out twos?

Mary Gail: Yes.

After the discussion revealed that all methods were the same, the students were asked to think about how a first or second grade child might solve this problem. The responses varied.

Lisa: Open each pack.

Renee: They could draw it and then draw a pack and this would be five and then draw the packs of 18, draw all 46 pieces out and group into twos.

Tena: Alright, so that would be an approach if they didn't have hands-on. Like Lisa said they could open up all the packs, if they opened up all the packs how do you think they would count?

Mary Gail: By groups of two.

Ann: Ones.

Tena: Lisa you think like one, two, one, two...?

Lisa: and then go back and count their piles of two.

The responses given were similar to the methods used by the students in the counting activities of lesson 1. While these methods would be appropriate for a small child, the students seemed to indicate their preference for a more sophisticated approach. They did not see the bubble gum problem as a counting problem but rather as a mathematics problem requiring an algorithmic process.

The party favor problem.

> Billy's mom is making party favors for his birthday. She plans to have sacks with candy and baseball cards to give each boy as they leave the party. She buys two packs with 15 cards per pack, three packs with 12 cards per pack, and one bargain pack with 54 cards. If she plans to put six cards in each bag, how many party favor sacks can she make?

This problem was given without the aid of a drawing or physical items. The numbers were chosen so that a drawing would become tedious.

Once again all of the students used the same approach. Mary Gail's description summarizes all of the approaches.

Mary Gail: Well, I did the same thing again. I took, uh, two times the 15 units, three times 12 units, one times 54 units, and added them together. Come up with 120, then divided by the number of cards that were to go into each bag which was six, and come up with 20 party favors.

Having Mary Gail verbalize her procedure made her aware of her dependence of units of one which was indicated by her comment "I did the same thing again" (Mary Gail). The dialogue continued:

Tena: Alright, so when you said I did it again, as far as unit, what did you focus on there?

Mary Gail: Uh, well, the outcome I broke into units of six, but, uh, asking for the original problem, I made it into individual cards.

Tena: OK, so again kind of like opening and dumping and pulling six out at a time?

Mary Gail: Right.

In an attempt to determine why the unitizing techniques of lesson 1 were not applied to the word problems the following discussion ensued.

Tena: OK, why do you think, like yesterday when everyone counted, everybody grouped... but when I gave you problems like this almost everyone of you switched back to units of one, where you kind of just dumped everything out and pulled.

Lisa: Easier, when you're counting hands-on there is more chance of a mistake because your mind wanders, things happen, so if you group everything it's easier to keep track of how many groups you have. In a math problem, when your doing it on paper I think it's easier to figure it out in ones because that's the way we've always done it. I mean, that's the way I learned to do it.

The comments made by Lisa in the above dialogue suggest that a possible barrier to connecting the unit structure to whole number problems was habit. This interference of previously taught methods and algorithms will be addressed in the second part of this chapter.

As the discussion continues Carolyn suddenly interrupts,

Carolyn: I was just sitting here and it occurred to me that you could just take the five sticks and add the 18 to it and then you would get the 23 because they're already broken into twos, two packs of five and two packs of 18.

Everyone seemed surprised. Ann commented, "that's true." Carolyn seemed pleased with herself and everyone laughed.

Carolyn's revelation is an interesting one. It was not until the discussion suggested a connection between the two lessons that Carolyn thought of using a unit other than one. When the connection was suggested she began to see the problem in a new light. It suddenly became more than a math problem on a page, it suddenly became a situation. Carolyn's previous focus on 2(5 stick unit)s and 2(18 stick unit)s was reconceptualized as 5(2 stick unit)s and 18(2 stick unit)s which suggests that she was constructing the inverse relation between the number of parts and the size of each part. This new focus allowed Carolyn to obtain 23(2 stick unit)s and therefore 23 players. This discovery shows real progress in her understanding of the unit concept and is consistent with one of the unit conversion principles identified in Behr, Harel, Post, and Lesh (1992). This principle involves the recognition of the equivalence between a(b unit)s and b(a unit)s.

The students were then asked if they connected the two lessons before the discussion.

Tena: Like thinking about how you did things yesterday and how you did things today, did you connect those or did you see any relationship or any similarities, before we started digging in and looking at units? Did you think about those as being alike in any way?

Mary Gail: Uh, not really. I saw today's task as more of a grouping project, whereas Tuesday I saw it as seeing how many one units there were. Today was how many groups of two or six there are, but today there was a lot more grouping for me, I know most of you grouped a lot Tuesday.

This was an interesting remark by Mary Gail since she also used a variety of unit structures during lesson 1, but converted everything to units of one in lesson 2.

Tena: But you grouped, like yesterday, like your group of a hundred. But you really weren't thinking?

Mary Gail: No, I didn't mentally think about it, Uh, I thought today was easier than Tuesday.

Mary Gail is still not aware of the unitizing she used in lesson 1. The dialogue continued.

Renee: I see this more as like addition, because you are trying to think how many can I put here, and I think, in all the problems I grouped, but I grouped everything on Tuesday. Today I was looking at how many of these can I put here and with the donut stuff, how many groups of three can I pull out of the box of 12. The idea of grouping was there both days for me.

After the discussion of units and grouping the students were asked to repeat the bubble gum problem and to try to focus on a different unit. Ann's second solution illustrates a maturing of the unit concept. She demonstrates her approach physically as she described it, "I took like the two packs of five and divided by 2," [She held both packs in her hand then dropped one of the packs to indicate division by 2. She repeated this process with the two packs of 18]. "This is the number of players," [Ann held up the hand that contained the packs]. "They each got a piece from this stack [extended the hand holding the packs] and a piece from this stack" [pointed to the packs that were dropped].

Mary Gail seemed to force the notion of unit in her second approach.

Mary Gail: I took each pack and divided it by 2, in other words I took a five pack and divided it by 2 and got two and one-half sticks of gum, and then I took the 18 pack, divided by 2 and got nine sticks of gum, so then I had 11 and one-half sticks of gum and then I multiplied it by 2 and got 23.

Tena: OK.

Mary Gail: It's original, I know.

Finally the students were asked for a second approach to the party favor problem. Recall that in this problem the students are essentially determining how many groups of six are in 2(15 card unit)s, 3(12 card unit)s, and 1(54 card unit). The group discussions appeared to increase the students' abilities to make connections with the unit concept as revealed by the unit structure of the second solutions described below.

Ann: You could see how many sacks you could make of each set of two packs, like the two packs of 15, which is 30, you could figure out how many sacks by dividing by six, you could make five...

Tena: OK, just from that set of cards...

Mary Gail: That's what I did.

Ann: and from the other you could make six and then 54 divided by six is nine and just add those individual sacks together.

Lisa: That's what I was thinking.

Mary Gail: That's what I did.

The unit concept was further examined.

Tena: If that had said, two packs with 18, if I had made that an 18. Could you have done something different?

Ann: Divided by 6 then multiplied by 2.

The above dialogue revealed Ann's sudden flexibility with unitizing. She was now beginning to see that the division by 6 could take place after all cards were totaled, or by totaling the number of cards in each set of packs, or if the 15 were changed to 18, each pack could be divided individually. Ann's awareness of the various unit structures that were possible in the party favor problem suggested that the concept of unit structure which was observed in lesson 1 had been expanded to the whole number problems of lesson 2. At this point, the other students did not exhibit this level of understanding.

At the close of the lesson the students were asked if every problem could be solved by units of one and also by units other than one.

Renee: It would depend on the numbers, because you really couldn't use units of two if you were working with odds, I mean you could...

Lisa: You would get fractions.

Renee: You could convert fractions to decimals.

Carolyn: ...and everybody hates fractions.

Renee: It's easier, I think, to just use ones.

<u>Discussion</u>.

My reaction to the success of lesson 2 was mixed. While the students' solutions to the first task revealed various unit formations, the flexibility of the unit structure was completely disregarded in the second and third tasks. In their initial solutions to these, all of the students used a typical school procedure involving multiplication, addition, and division. It was somewhat disturbing that the awareness of units that brought excitement to the students in lesson 1 was now absent.

The explanation for the use of different units in task 1 appears to lie in the format of the problem. This task was accompanied by a diagram in which the physical characteristics of the problem were visible. This was not the case in the second and third tasks. It would seem that the presence of the diagram suggested a more simplistic solution whereas the others suggested a more mathematical, and thereby a more complex approach. When confronted with why the traditional approach was chosen over an approach indicative of the unitizing of lesson 1, the students' underlying reason was habit. These obstacles to the implementation of the unitizing concept will also surface in the remaining lessons.

Lesson Three

The previous lessons were intended to examine the students' formation of units in counting activities and whole number word problems. Lesson 3 was designed to further emphasize unit formation by providing situations in which whole numbers could be conceptualized as composite units. The students were given three tasks. In each task the students were asked to provide different arrangements of a whole number of items.

<u>The soccer ball problem</u>.

The intent of the first task was to allow the students to view a whole as composed of smaller units which represents a reunitizing of a composite unit. Cubes were provided for manipulation but the students were asked to draw their final arrangements on the task sheets. The problem was presented as follows:

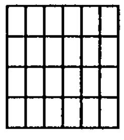
> Peter is trying to set up a display of soccer balls in his athletic shop. He has 24 balls to display on a table. The balls are in boxes to make them easier to arrange. For his first attempt, Peter tries 4 groups of 6 balls as shown below, but he doesn't like this arrangement. Please help Peter set up the display by showing him three more ways the balls can be displayed.

Peter's first try: □□□□□□ □□□□□□ □□□□□□ □□□□□□

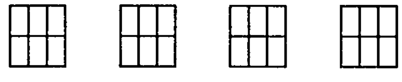
All of the arrangements given by the students were either in the form of a rectangle or some type of pyramid. The language used in their descriptions not only revealed the conceptualization of units but also the conceptualization of 24 as a composite unit. Mary Gail described, "two groups of 12 that were three by four." The fact that Mary Gail felt the need to describe the size of each group of 12 as "three by four" suggests that each group of 12 was considered one entity or one unit. Her conceptualization of the 24 balls could be described as 2(12 ball unit)s where each of the 12 ball units consisted of 3(4 ball unit)s or 4(3 ball unit)s. Thus, her conceptualization of the 24 balls seems to have a unit of units of units structure as (2(4(3 ball unit)s-unit)s-unit).

In describing her pyramid, Lisa stated, "I did one at the bottom which was 10, and then eight, and then four, and then two." Lisa's use of the word "one" supports the contention that each row of the pyramid was considered to be a separate unit yielding a unit interpretation of 1(10 ball unit) + 1(8 ball unit) + 1(4 ball unit) + 1(2 ball unit).

Carolyn's arrangements also revealed unit formations. She stated, "I had four groups of six [gestures to indicate one thing], and four groups of six [gestures to indicate four different things]." While Carolyn's description sounded identical, the drawings on her task sheet revealed the difference.



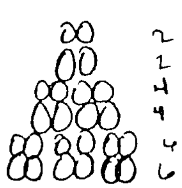
(Four groups of six)



(Four groups of six)

In her first arrangement the "group of six" was used to refer to one row of the rectangular grid. Whereas in the second arrangement the "group of six" was used to refer to a whole grid. The unit interpretation of Carolyn's first arrangement could be described as (1(4(6 ball unit)s-unit)-unit) which indicates the unit of unit of units relation and the second as 4(6 ball unit)s which is a unit of units.

While most of the arrangements were similar to those above in that each arrangement was described in one way, Renee's last arrangement was unique in that she provided three descriptions of the same arrangement. In describing her last arrangement Renee states, "The last one started out as six groups of four, but as I started stacking across I realized they could go together as a pyramid so it would be six, six, four, four, and two, two." In this one arrangement Renee conceptualized the balls as 1(24 ball unit) then as 2(6 ball unit)s + 2(4 ball unit)s + 2(2 ball unit)s and finally as 6(4 ball unit)s. This is clearly seen by the drawing on her task sheet.



6 groups of 4 =
6 × 4 = 24

After the students had described their arrangements some of the unit formations were written on the board using the conceptualized unit notations (i.e., 4(6 ball unit)s, 2(12 ball unit)s, etc.). As the units were discussed the students were probed:

Tena: Look at all these different units that we've used...do you see a relation between the size of the unit, these are all the different sizes we used [refers to the board] and the number of units we needed of that size. How do those relate, do you see any relationship there?

Carolyn: They're all factors of 24.

Tena: OK. They are all factors of 24. As we make our units bigger, what happens to the number of units that we need?

[Students answer in unison]: Gets smaller.

The dialogue above reveals that at the end of the first task the students were beginning to view a composite unit as a quantity whose magnitude is determined by the unit structure imposed upon it which depends on both the size of the unit and the number of units.

<u>The table problem</u>.

The second task provided a natural setting for looking at a particular situation in terms of various unit formations. In this task the number of units was supposedly established by the two tables but the size of the unit was to be controlled by the student. The task was presented as follows:

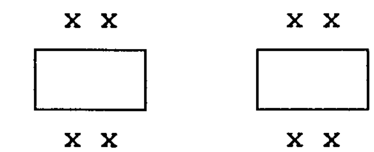
> Eight people must sit at the two tables below. Chairs are stacked in the corner of the room so each person must get a chair and take it to one of the tables. Please indicate three possible seating arrangements in the space below.

While it was predicted that the students might have arrangements like five at one table and three at the other table, this was not the case. All students used some arrangement of four at each table. The only arrangements that differed from this were those where the students pushed the tables together and had one group of eight. Again this was not expected. It was thought that the number of units would be controlled by stating the existence of two tables.

In discussing their arrangements the students' comments demonstrated their unit formations as well as their struggle with the unit terminology.

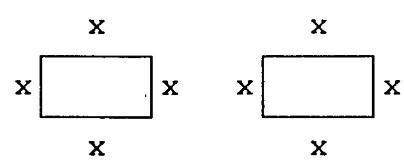
Tena: OK, if we look at groups and the unit idea like we did while ago, what did you do in each one of these?

Mary Gail: OK, the first would have been like two units, with two on each side, two, two units of two on each table [refers to the diagram that follows].



The conceptualization of the above arrangement as indicated by Mary Gail's description is (2(2(2 chair unit)s-unit)s-unit) which suggests the unit of units of units relation.

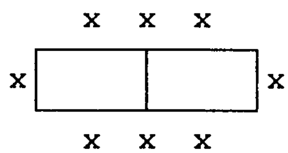
Mary Gail: On the second one I would say, I would look at it and say singles or you could say two groups of four, but I see it as one units, four one-units at each table [refers to the diagram below].



The unit formation in Mary Gail's second description can be noted as 4(1 chair unit)'s.

Two of Lisa's arrangements involved pushing the two tables together to make one large table. She describes one of her methods below.

Lisa: My second one I put the two tables together to make a long rectangle and had three on each long side and one on each end [the diagram below corresponds to the diagram Lisa had drawn on her task sheet].

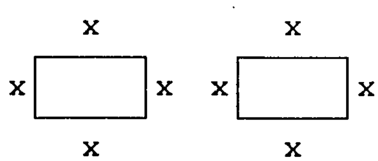


Tena: If we look at units here, how could we describe that one?

Lisa: A unit of eight people.

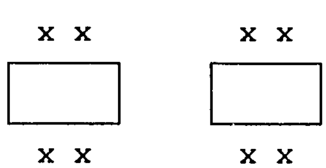
Knowledge of the nesting characteristic of units was demonstrated in Ann's description that follows:

Ann: But I don't really look at it as singles [refers to the diagram below]. I see two groups of four,



On her task sheet Ann had written "2x4" beside the above arrangement which further supports the conceptualization of 2(4 chair unit)s.

Ann: and this one, I see two groups of four with two groups of two within.



In the above description the use of "within" suggests that Ann was recognizing the nesting characteristic of the unit concept. This is further supported by her task sheet where she wrote "2(2x2)" beside the above arrangement. This notation seems to support the conceptualization as 2(2(2 chair unit)s-unit)s.

After noting in the discussion that everyone generated the same number of people per table, the students were asked for another approach:

Tena: If I said you can't have the same number at every table, what would have been another way that you could have done it?

Renee: You could have done six and two. Like two on the ends.

Tena: If we had done that, what would have been the unit structure?

Renee: Six and two.

Tena: OK, 1(6 unit) and 1(2 unit). Is there another way we could have done it?

Lisa: Five and three.

Tena: OK.

Lisa: All at one table.

Tena: OK, we could crowd all of them around one table and leave the other table empty.

Carolyn: Or put seven and the other person nobody likes at the other table [laughs].

In discussing the use of four per table, the students were introduced to the term norming as the tendency to break groups into the same amount. Lisa responded to this notion:

Lisa: I was thinking what would look best.

Tena: OK, more pleasing to the eye, maybe, OK.

Lisa: Two tables of four looks better to me.

Carolyn's final response was indicative of a pattern that was noticed as the teaching experiment progressed, the more realistic the problem appeared the more flexible the unit formation.

Carolyn: If you just had people coming in randomly of course they are going to sit by someone they know, so they might not work it out the way you want.

The faculty meeting problem.

In the final task of lesson 3 the students were given a task in which they were asked to norm a set of composite units.

> After the faculty meeting the lecture hall was a mess. There were 2 tables with 6 chairs each, 3 tables with 4 chairs each, and 2 tables with 2 chairs each. Please straighten the room so that there are the same number of chairs at every table. Please draw you solution in the space below.

All of the students used a unit of one approach. They determined the total number of chairs and the total number of tables and then divided.

Mary Gail: Well, I added the number of chairs together. There were two tables with six chairs, so that is 12 chairs, and there were three tables with four chairs, that's 12 chairs and then two tables with two chairs. You added them altogether and you came up with 28 and I was supposed to divide by the number of tables, which was seven and then you came out with an even number, four.

The other students described a similar process. Then the students were asked to model the problem by using their task sheets as tables and using the cubes as chairs. The students laid out seven task sheets and put six cubes on two of the sheets, four cubes on three of the sheets, and two cubes on the remaining two sheets. The discussion continued:

Tena: OK, if you were the janitor walking into this room would you pull all of the chairs away from the tables, which is really what you are doing when you convert to singles?

Renee: No.

Tena: What would you do if you were the janitor?

Renee: He would know ahead of time that they all needed four chairs so he would probably go...[she begins to move the cubes].

Tena: OK, what if you were a student that really didn't know ahead of time, you just pulled someone out of fifth period study hall and said please go straighten that room before the next meeting and didn't really have prior knowledge.

Mary Gail: Start pulling chairs to other tables and start evening out [models physically by moving cubes]. I would take, I would take one from each table first and give it to the tables of two and then, you know.

Tena: So you would pull from the biggest tables first?

Mary Gail: Pull from the biggest and take it to the smallest and see how even they were and then..

Tena: OK, do that.

Mary Gail demonstrates physically how she would move the chairs to end up with four per table. Then the dialogue resumed:

Tena: OK, so in a realistic setting you wouldn't take away all the chairs, stack them up and start distributing one chair at a time, like we really did mathematically. What was the unit here that we focused on?

Mary Gail: Four.

Tena: OK, we focused on four pretty much immediately rather than one and then divided.

Ann: You could have counted all the chairs before you started to do it and then divided by four.

Mary Gail: You could, but what if it were like a big conference room?

Renee: I would probably walk in and look at the average, like how there were three tables of four.

Tena: Alright.

Renee: I would try to visualize it instead of going directly to the small tables, it would probably, for me, I think it would save my time in case six was supposed to be right or something. It's just when you're glancing over a room you don't really go one, two, three, four...when you're just looking. You look at clusters. So I would look at four and say that would be the average for the table.

Ann: I still think I would count 'em.

Tena: You still think you would count?

Ann: I'd have to do that.

Lisa: I think I'd count them, but it depends on how many tables you had.

The above discussion provides a more realistic view of the task which was originally seen as just another math problem. This discussion provides further evidence that the more realistically a problem is viewed the more flexibility in the unit approach.

Lesson Four

The fourth lesson involved a group task in which the students were in complete control of the unit quantity. Previous tasks had fixed either the size of the unit or the number of units. In this task the students had to decide on the size of the unit and also the number of units. Due to the nature of the task, a unit of one approach was not realistic. The situation presented in this task forced the students to consider an overall norm among groups and/or the norm within groups. Since there were various solutions to the problem the researcher was interested in the process used in determining a norm and the unit structure for a particular solution. This task was presented as a group activity in hopes that an open discussion of unit structure would challenge the thinking of some of the students and affirm the thinking of others.

The prom problem.

The task read as follows:

> The junior class is busy planning for the prom. The decorating committee is trying to decided on the number of tables it needs. There are 225 seniors but not all the seniors are planning to attend. According to the latest count there will

> be 56 couples, 15 groups of four people, 8 groups of three people, and six people are coming alone. Please decide how many and what size tables are needed. Then explain how you would arrange the name tags.

The students were reminded that since they must leave room for the dance floor they probably needed to order as few tables as they could. The discussion began.

Mary Gail: Well, let's figure out...

Renee: How many people, total.

Mary Gail: But keep it separated.

Everyone began to calculate and compare to make sure they have the same number of people. Then Ann interrupted with an important reminder,

Ann: It really doesn't matter how many total people you have, as far as tables, because we want to keep them together.

This remark by Ann along with Mary Gail's suggestion of "keep it separated" suggests the realization that a unit of one approach was not appropriate. They continued their discussion.

Carolyn: We could take one big table and put the six stags together.

Lisa: What if we put six people to a table?

The suggestion of putting six people to a table represented their first attempt at finding a norming unit. They began to mumble and work silently as they individually toyed with the idea of six per table. As they began to have difficulties the discussion resumed.

Mary Gail: OK, so there's one table for the singles.

Lisa: And eight groups of three people so put two groups at a table...

Mary Gail: Eight groups of three people is...

Lisa: Four tables.

Mary Gail: 24 people, and you are having tables of six? So, four.

The dialogue between Mary Gail and Lisa suggested that they were on different levels in regard to the understanding of the unit concept. Lisa was quickly able to determine that eight groups of three people would be four tables or 4(6 units). Mary Gail had not reached that stage. She converted the eight groups of three people to individual people and then divide by six to get the four tables. Lisa sensed this lack of understanding and adjusted her next explanation.

Lisa: 15 groups of four is 60 people so, that would be 10 tables...

Renee: OK, well you can have one two-people sitting, you can get the nine six-people tables out of that, you can have nine tables and have a small two-people table.

The difficulty of the task was realized as students began to raise some realistic issues. This indicated that the task began to be considered as a realistic situation and not as a routine math problem.

Mary Gail: Well, not everyone is going to be sitting at the same time, are they? Do you eat dinner at a prom?

Lisa: It depends on what kind of prom.

Carolyn: Everyone will be walking around, mingling, dancing, getting punch, juice, so they might not be all sitting at the same time.

Ann: If we had 33 tables...

Renee: But we just need to say in case they do sit down they could all be at a table.

While the social activities of a prom were discussed Ann was still working on norming by six. She interrupted the discussion,

Ann: You could have 33 tables that seat six people and one left over that seats four people.

Ann's task sheet revealed that this conclusion was obtained by dividing 202 by 6. This yields a quotient of 33 and a remainder of four. At this point Ann seemed to be disregarding the units that had to remain intact.

Renee had not determined a solution but she did recognize that Ann's solution would not work, "But then you're still looking at, you are going to have some of these broken up [points to the description of the groups]." The group seemed confused and was having difficulty with norming by six. Carolyn suggested another approach.

Carolyn: What if we put eight, eight at that table? Then it might help with the other six.

Lisa: So six and a couple would be eight...

Mary Gail: If you had tables of eight...

Again the students began to mumble and compute individually on the new norm of eight. At this point Renee left the discussion and was busy drawing out her solution. When the discussion resumed everyone seemed to have a solution.

Mary Gail: OK, I think we should divide the tables into four-tops, six-tops, and eight-tops. A four-top meaning a table that seats four. So for the people who wanted to remain in groups of four you would need...OK, well the six-tops you will need one for the groups alone and, one for the ...or five six-tables. For the tables of four you would have 15 fours and then the 112 divides evenly by eight, so you could have 14, 14 eight-tops.

In Mary Gail's approach she did not norm by a single unit but rather by fours, sixes, and eights. By this time Ann and Renee also had workable solutions. Renee's method used fewer tables and therefore became the focus of the discussion.

Everyone seemed surprised at Renee's method. They had watched her drawing the tables but the process had taken some time. They questioned her to make sure the various groups were intact. Mary Gail suggested that she had counted wrong and the total tables was actually 23 rather than 24. They all agreed and moved to the next part of the task.

Lisa: I think that's good. How would you arrange the name tags?

Having the students place name tags as part of the task was just a means of assuring that the groups would stay together. The students took this part more seriously than I had anticipated. Again the realistic nature of the problem surfaced.

Carolyn: Well, we could just put couple, couple, couple, single, single, [pretends to place tags on imaginary tables]...

Renee: Yeah, but a lot of times couples could sit across from each other...

Ann: I like her idea of just putting couple, couple...single...

Renee: Why don't we just write loner...

Carolyn: Nerd...

After much discussion on the problems with arranging the name tags the students made a final adjustment on their original solution. They decided that a better solution would be 1(12-top), 10(10-top)s, 9(8-top)s, and 3(6-top)s. This arrangement used the same number of tables but provided a more socially correct way to arrange the name tags.

After the discussion of the prom task the students' sensitivity to the unit structure was probed by asking them to respond to adding 12 inches plus three feet. Initial responses were "four feet" (Mary Gail) and "three feet, 12 inches" (Renee). When asked if they could do it another way the unit structure became more flexible.

Lisa: Four 12 inch units

Ann: Or you could go 12 one inch units and 36 one inch units...48 one inch units.

As the above dialogue continued we see that the students were becoming more sensitive to the notion of unit and were beginning to become more flexible in their unit structure.

Tena: What if I just give you two whole numbers and kind of the idea that you have to find something common about them. Like before you can add 12-inch units together very easily. If I ask you to find something about those so that you are not adding six one-things plus four one-things, how could you rewrite that six and rewrite that four?

Mary Gail: They are both divisible by two so three two-units and two two-units, gives you five two-units.

Tena: Which converts to what if I go back to ones at that point?

Mary Gail: 10 units.

As the discussion continued the students seemed to be examining some previous conceptions:

Lisa: I think I think that way, you know when you say it I know I think that, I know my brain goes through that process, but when you ask me what I think I just say six plus four is 10, six units plus four units is 10 units, but I don't realize I break it down as much as I do until I see that 3(2 units). I guess I really do do that, I mean, but if you asked me to explain it I wouldn't tell you that way, I'd just say the simplest way which is six plus four is six units plus four units is 10 units.

Renee: We are programmed when we are younger to just memorize it [refers to 6+4=10], you know you look at this as 10.

Mary Gail: I was wondering about that in units of 10. I don't think of 10 ones, I think in groups of 10. If someone said add one thousand and a hundred together, I don't think of a thousand one units, I think of a 100 10-units.

Tena: So even if you had 20 + 30, you think mentally you think in 10 units here?

Mary Gail: Right.

Lisa: ...a thousand is one thing, a hundred is another unit, ten is a unit. If you're adding a thousand and a hundred, that's one one-thousand and one one-hundred. Not one hundred units, but one unit of 100, and one unit of 1000.

The above dialogue suggests that the students were beginning to connect the unit concept with their previous knowledge. In particular, Lisa and Mary Gail noticed the connection to place value.

At this point in the discussion, the students were asked to respond to a child who gave an answer of eight when asked to add six inches plus two feet.
The responses reveal the students' increasing sensitivity to the unit concept.

Carolyn: He is not looking at feet and inches.

Mary Gail: He is not looking at the format of the unit, you know, whether it is in feet or yards, it had to all be the same unit.

Ann: He is not looking at the size of the unit, he is saying two of something is equal to...

Lisa: It is the same thing in algebra. You have an x and a y and they tell you that you can't add apples and oranges the same. You can't add apples and oranges. So basically you have to explain to him that an inch is not the same as a foot so you can't put those together.

Ann: They are different values.

Lisa: Right. That's a good word for it, different. And you have to change one to add them both together. I can't explain, you know what I'm saying, you have to change the foot or the inch to equal the other, the value of the other one so you can add them together.

Lisa has found another connection between the unit concept and her previous knowledge. In the above dialogue she associates the algebraic process of combining like terms with the process of adding feet and inches. She seems to be conceptualizing the importance of the unit but she has difficulty putting her thoughts into words. The continued discussion of the students suggested a possible remedy for problems like adding feet and inches.

Renee: It reminds me of when we did word problems in like elementary school, and if you didn't write, like if you just put the answer and you didn't write something behind it, you always got in trouble. They tried to tell you what is this, you have to say 12 of what.

Lisa: It makes you think though, because you have to do that, and you do it like that. I make mistakes, I still make mistakes with that, you know, six plus two is eight and if I go to write eight, eight inches, I say wait a minute that's wrong.

In the above discussion, Lisa and Renee revealed that having to provide more than just a numerical answer, that is, the number of units as well as the type of unit (e.g., 12 inches), increased their sensitivity to the unit structure. They continued to discuss the importance of this in regard to children.

Renee: It starts making them watch...

Lisa: For what a unit really is...

Renee: Pay attention to what they are really adding.

Lisa: Like six balls plus four balls, each ball is one unit. Then if you had six balls and two...

Ann: packages of softballs...

Lisa: Right, or two tennis rackets or something. I mean, you can't, you know what I'm saying, they have to realize that. If you give something like that, they don't really understand what a number is, they know a number, but they don't understand...

The discussion of lesson 4 revealed some of the students' ideas for helping children understand the importance of the unit. They suggested that when children first begin to add, they should be given problems with the unit labeled. In addition to the labeling, they also suggested that students should be given problems containing different units so the students must find a common unit to work the problem. Lisa suggested that an early approach to this type of problem is to draw or model the situation. It is my contention that the above suggestions, although intended for elementary school children, revealed what the students deemed necessary for overcoming some of their own obstacles in cognizing the concept of unit.

Lesson Five

Lesson 5 represents the first lesson in which fractions were addressed. This lesson consisted of two tasks. The first task was designed to provide a situation in which a unit was used as an actual measuring device. The size and types of units in this task necessitated the introduction of fractions. In the second task the students used fraction pieces to extend the notion of number as a composite unit to the rational number domain.

The measuring task.

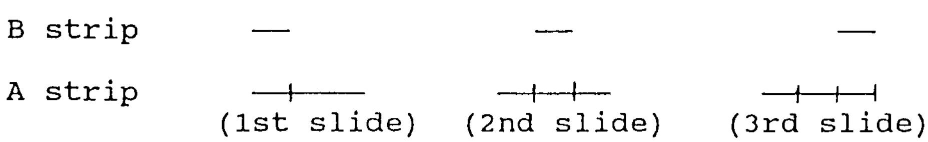
In the first task the students were given three cardboard strips to use as measuring units or rulers. The students were asked to measure each one in terms of the other two (See Appendix B). This activity was designed to enforce the concept that a measurement depends on the unit used to measure (i.e. the unit used as the "ruler"). As the task progressed the measure of each cardboard strip took on a different value as the measuring unit was changed. By measuring the small strip (B) with a larger one as the ruler (A or C) the students had to inject the notion of fractions. This denotes the first time that fractions were addressed in this teaching experiment. By measuring a larger strip with a smaller one as the ruler, the strip previously labeled as a fraction becomes the new unit by which to measure. This encouraged the students to consider the unit fraction as a whole (a unit) which could be iterated to determine the measure of a larger strip.

An object can be measured by comparing the ruler to the object or by comparing the object to the ruler. To provide experiences with both approaches to measuring, the smallest unit (B) was attached to the table and the other two units were loose. This setup provided situations in which the ruler was manipulated as well as situations in which the object was manipulated.

On the first problem in task 1 the students were asked to measure the smallest strip (B) using the medium strip (A) as a ruler.

B —— A ————

All students proceeded in the same manner: a measure and count approach. The medium strip (A) was slid alongside the smaller strip (B) as they counted. Carolyn described her process on the task sheet, she writes, "Put A-unit next to B and marked where B ended on A. I did this three times."



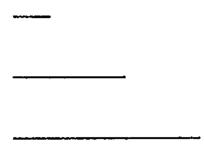
Renee's process can also be described by the above diagram. On her task sheet Renee writes, "I compared A to B to see how many "B"'s [sic] there were in A. There are 3 B's in A. Unit B is the legnth [sic] of 1 part of A. Thus, 1/3." It is also interesting to note that after Renee determined that the measure of B was 1/3 using A as the unit of measure, she took the A strip, folded it into thirds, and then compared it to the B strip as if to check her answer.

The second problem asked the students to use unit B (small) as a ruler to measure unit A (medium) and unit C (large).

B

A

C



In reference to unit A the following dialogue occurred:

Mary Gail: OK, to get, uh, one A unit to equal how ever many B units, I figured up that there were three B units that make up one A unit.

Tena: OK

Mary Gail: Uh, that's from the previous exercise. It is the reciprocal of one third.

Tena: OK, so you really didn't measure anything, you just kind of used what you found out in number one?

Mary Gail: Right.

Tena: OK. Did anybody else do it differently?

Lisa: I did the same thing.

Carolyn: I did too.

Tena: Did anyone actually measure?

Renee: I just looked at A and whenever I looked at A I saw the three.

It should be noted here that "the three" Renee refers to is the three creases that were still visible from when she folded the A unit (medium) into thirds to check the measurement of unit B (small). The dialogue continues.

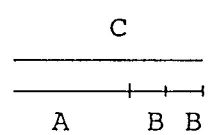
Mary Gail: I think I measured everything just to check, since this was fractions.

Ann: I still went one, two, three (refers to the sliding process).

In determining the number of B units (small) in a C unit (long), Carolyn was the only student who used an approach other than slide and count. Her process reveals the conceptualization of the C unit (long) as a composite unit.

Tena: OK, did anybody do anything different?

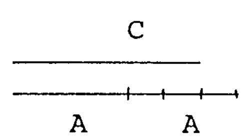
Carolyn: I did. I put the three units in my A and marked them down, then I brought it down and found two more.



Tena: OK, so you like looked at C as one A plus...

Carolyn: two B units, yeah.

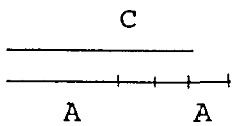
It was on the last part of problem two where the uncertainty with fractions began to surface. The students were instructed to use unit B as a ruler to measure unit A and unit C. Then they were given the statement 1(C-unit)=_______(A-unit) and asked to fill in the blank. As the students proceeded to do this task individually there seemed to be no confusion. However, an analysis of the task sheets revealed that Mary Gail and Carolyn seemed to be unsure of the correct answer. Mary Gail indicated her uncertainty by putting two different answers beside the blank (1 2/5 and 1 2/3). Carolyn's sheet showed numerous erasures to indicate that she too was uncertain about this. The fact that the others seemed to show no doubt in their answer might stem from the choice of the unit used to measure the C-unit. Renee's task sheet revealed her method, she wrote, "I found 1 A in C then I made A into thirds and found 2/3 in C, thus 1 2/3."



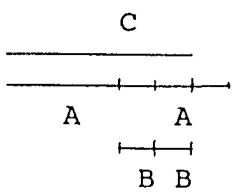
Recall that in Renee's very first task she folded the A-unit (medium) into thirds to check the size of the B-unit (small). The visible creases that remained after this folding helped Renee in measuring the C-unit (long).

Mary Gail became confused when the A-unit (medium) did not fit beside the C-unit (large) an even number of times. Mary Gail wrote, "There is 1 whole a-unit and a

<u>part</u> of another a-unit. I already had a and c divided into B-units so the part left over on a = 2/5 B-units."



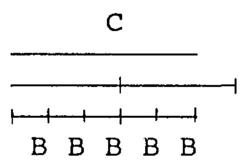
Mary Gail introduces the B-unit (small) to help her find the " <u>part</u> of another a-unit" (Mary Gail). At this point Mary Gail has switched the ruler to the B-unit (small). The fractional part of the A-unit (medium) is equivalent to two B-units.



She has previously determined that there are three B-units in the A-unit and five B-units in the C-unit. Her difficulty arises in deciding whether the two B-units represent two fifths or two thirds. Mary Gail's answer of two fifths indicates that she was using the C-unit as the ruler, in other words, she was confusing the unit to be measured with the measuring unit.

The numerous erasures on Carolyn's task sheet suggested that she was also confused. However, she appeared to resolve her confusion as indicated by her final answer and by her description of the process.

Carolyn wrote, "I measured the A unit next to the C unit there were 3 B-units w/ [sic] 2 B-units left."



(Puts A-unit next to C twice)

(Translates to B-units)

As with Mary Gail, Carolyn had to interpret the five B-units. The erasures indicated that Carolyn had first made the same mistake as Mary Gail. She had first written "1 2/5" in the answer blank which indicates her initial interpretation of the five B-units also used the C-unit as the ruler.

It is interesting to note the mix of unit structures in Carolyn's and Mary Gail's solutions. Carolyn indicates there are five B-units but she first interprets these five B-units using different units of measure. Three of the B-units she calls one A-unit (as indicated by the whole number portion of the 1 2/5 answer that she first gave) which would suggest that each B-unit was considered one third. The remaining two B-units were interpreted as two fifths which suggest they were measured against the C-unit. This was also Mary Gail's interpretation as indicated by her answer of one and two fifths. Something caused Carolyn to erase the one and two fifths answer and change her answer to one and two thirds. While one can only speculate, perhaps Carolyn realized that she was using two different rulers in determining her final answer. If the same ruler was used the five B-units would either have to be interpreted as five one-fifth units (if C were the ruler) or five one-third units (if A were the ruler).

The confusion shared by Mary Gail and Carolyn which was visible on the task sheets was broadened during the discussion. The following dialogue indicates the uncertainty of the other students.

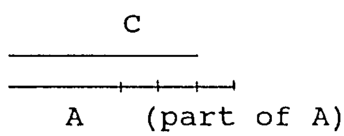
Tena: Alright, now the last one on number two, you had to put C in terms of A. How did you do that? Start with Renée.

Renee: I looked to see what was in A and then I marked what A was and then I could see from measuring earlier that there were two parts left and I remember I brought it down to make sure it was the same (compares to the C strip) so then I knew it was one and two thirds, I mean two fifths, two thirds.

While Renee's task sheet suggested she understood this problem the above dialogue shows otherwise. As she explained her method she visibly became confused. When she stated "I brought it down to make sure" (Renee) she refers to measuring the B-unit against the C-unit. At this point she is in the same boat as Carolyn and Mary Gail. The dialogue continues:

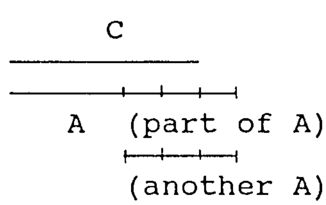
Mary Gail: I saw everyone's paper, not that I'm spying on everyone, but I thought, uh, oh, I had two thirds and then I switched to two fifths.

Lisa: It is two thirds because A is in thirds and that is one A and that is two thirds....Right?



Lisa began with a lot of confidence but the silence of the other group members caused some doubt. For a moment there was silence as everyone seemed to be rethinking the problem. Carolyn broke the silence with confidence,

Carolyn: See if you had one more little piece, you would have another A.

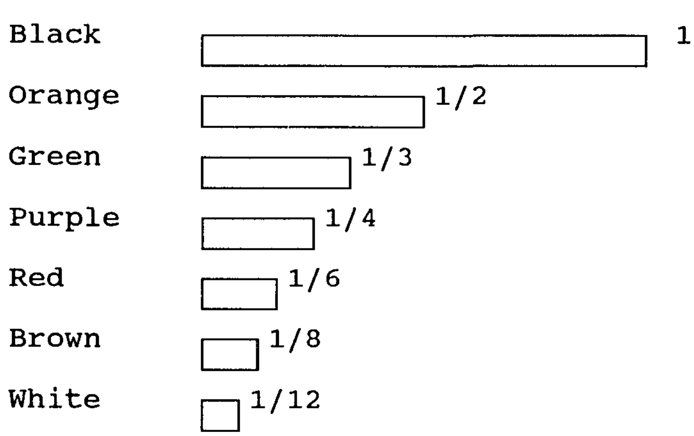


Carolyn's observation that "one more little piece" would generate a whole A-unit appeared to solve the dilemma. The others seemed satisfied and we proceeded to the next task.

<u>The fraction pieces</u>.

For the second task the students were given a bag of colored rectangular pieces that represented different fractions. This part of the lesson was not meant to be an introduction to fractions or a review of fractions but rather it was intended to provide additional opportunities for the students to use a unit fraction as the actual measuring unit.

Before beginning the second task the students were asked to find the measure of all the different colored pieces by using the black piece as one unit. Figure 2 illustrates the relationship among the different pieces.



<u>Figure 2</u>. Size and color of the fraction pieces

The procedure used by all of the students was the same. They first modeled the unit, covered the unit using a particular colored piece, and then counted the number of pieces to determine the size. On the back of their task sheets each student recorded the size of each piece in terms of a unit fraction (i.e. black = 1, green = 1/3, orange = 1/2, etc.).

Once the measure of each piece was described the discussion provided opportunities for the students to conceptualize a unit fraction as a composite unit.

Tena: OK, show me the one-half piece. If we look at that as a unit, that is one one-half piece or one one-half unit. If I wanted to give another name for that as a composite unit or a group of units, what other color could I use to rename that one-half piece?

Renee: Two purples.

Tena: OK, so one one-half unit would be the same as two purples or two, what? What could I call those?

Ann: Two one-fourths.

Tena: OK, so that one one-half unit I could think of as two one-fourth units. OK. Is there anything else you can think of for that?

Carolyn: Three reds.

Tena: OK. If I go to the unit idea that would be three...

Renee and Carolyn: Three one-sixths.

Tena: OK, Three one-sixth units.

Ann: or four one-eighths.

Lisa: Six whites.

It should be noted that when the students were first asked to provide another name for the one-half piece, they actually took pieces and covered the one-half to determine their answer. As the discussion progressed they stopped solving physically and began to answer quickly. This might be suggestive that some of the students recognized their earlier rote means of responding to matters involving the notion of equivalent fractions. However, it could also be argued that this was not the case since answers were given as three one-sixths, four one-eighths, and six whites rather than as three-sixths, four-eighths, and six-twelfths.

Responses such as "three reds" (Carolyn), "six whites" (Lisa), and three one-sixths (Renee) indicated that the individual colors that represented a unit fraction were being conceptualized as wholes or units. These responses indicate an important step in cognizing the unit fraction as a unit that can be replicated and united to form a composite unit (e.g., three-sixths was considered as three iterations of the one-sixth unit).

The notion of composite units was further expanded as the students were asked to use two colors to rename a given piece.

Tena: Could you name it using two colors? You've told me all browns or all reds, find me a way that would use two colors.

Carolyn: One purple and two browns.

Lisa: Three one-twelfth pieces and two one-eighth pieces.

Tena: OK. Ann what have you got?

Ann: One one-third unit and, uh, One one-sixth unit.

Carolyn: One one-sixth, two one-eighths and one one-twelfth unit.

Renee: I have one one-fourth, one one-twelfth and one one-sixth.

Tena: ...and Mary Gail, you've got...

Mary Gail: Oh, I don't have anything, I've just been thinking. I can say two one-eighths and three one-twelfth units.

The above dialogue reveals that students were becoming comfortable with the composite nature of a unit fraction. The second task was designed to extend this notion to common or non-unit fractions.

> Using the fraction pieces, please describe four different ways to illustrate the following rational numbers as composite units. Please use the format a(b-units) where a indicates the number of units and b indicates the size of the unit.

a) 2/3 b) 3/4 c) 5/2 d) 5/6 e) 4/5

Again the students used the strategy of model, cover, and count that had been used previously when the students were asked to determine the size of each piece. Renee's strategy had a slightly different twist. In my observation journal I wrote, "Renee always compares to the basic black unit. For 3/4 she uses 4/4 lined up against the black and then covers 3/4. Instead of modeling 2/3, she uses 3/3 and then covers 2." This dependence on the black unit can also be seen in her responses on the task sheet. For one of her descriptions of two-thirds Renee writes, "4(1/6 units) on 3(1/3 units)." For one of her descriptions of three-fourths, Renee writes, "3(1/6 units) + 1(1/4 unit) on 4(1/4 units)." This strategy began to confuse Renee when she tried to describe five halves. She indicates her confusion by writing a question mark beside this problem on her task sheet. Her four attempts at describing five halves were all unsuccessful, Renee writes, "5(1/12 units) on 2(1/2 units)" and "2(1/6 units) + 1(1/12 unit) on 2(1/2 units)" (Renee). It is interesting to note that all four of Renee's descriptions were models of five-twelfths.

? c) 5/2 5(1/12 units) on 2(1/2 units)
2(1/6 units) ~~=~~ + 1(1/12 unit) on 2(1/2 units)
2(1/9 units) + 2(1/12 units) on 2 1/2 units
1(1/3 unit) + 1(1/12 unit) on 2 1/2 units

My first reaction was that perhaps Renee misread the problem as five-twelfths instead of five-halves, but this theory has no support from her choice of units. By this I mean that when she modeled two- thirds, she indicated that her models were "on 3(1/3 units)" (Renee). On her models of three-fourths Renee indicated "on 4(1/4 units)". If Renee had truly misread five-halves as five-twelfths her strategy would indicate that she should have "on 12(1/12 units)" instead of what she actually had which was "on 2(1/2 units)" (Renee). I became further confused by her apparent descriptions of five-sixths. By drawing the line on her task sheet I assumed that the descriptions beside five-sixths were intended to be descriptions of five-sixths, but this doesn't appear to be the case.

d) 5/6 5(1/2 units) on 6(1/2 units)
10(1/4 units) on 6(1/2 units)
30(1/12 units) on 6(1/2 units)
5(1/3 units) + 5(1/6 units) on 6(1/2 units)

Each of these descriptions are correct models of the five-halves. Which might indicate that by drawing the line she started anew on the problem of representing five-halves and completely disregarded the five-sixths. These questions concerning what Renee actually meant were never answered. Due to the time constraints of the lessons, this task had to be interrupted before all of the students were finished which left very little time for discussion. The only discussion was as follows:

Tena: OK. I think most of you are almost done. I promised...I would be through at twelve. Lisa, it looks like on the last one, it looks like you stopped modeling and started doing it in your head?

Lisa: Well, there was no piece I could use for fifths, so I went ahead and started doubling the fractions, four-fifths, eight-tenths, ten-twentieths...

Tena: Ann, what about you on the last one?

Ann: I just multiplied four-fifths times 2 and got eight one-tenth units, times 3 and got twelve one-fifteenth units, by 4, and by 5.

The above discussion reveals that Ann and Lisa used the notion of equivalent fractions to get various composite units for four-fifths since there was not a fraction piece that corresponded to one-fifth. The other students did not get to this problem. The discussion continues.

Tena: Can you see any advantages for a child to be able to look at two-thirds in many different ways?

Mary Gail: Oh yeah.

Renee: I think it's better to convert something, like, it would be easier for them to say, like, or then, if I have one-half that also equals two one-fourths or four one-eighths and it would be easier for them to see.

Tena: OK.

Renee: Cause usually they will see one-eighth and say well one-eighth is bigger than one-fourth, just look at the eight is bigger than four.

Tena: Why do you think they do that?

Renee: Just seeing the bigger number.

Tena: You think just because, normally eight is bigger than four, so when they look at one-eighth and one-fourth they go back to what they knew before?

Lisa: Yes.

Tena: Do you think focusing on one one-eighth piece rather than one-eighth would make any difference?

Lisa: Yes. You could see the small piece of the original whole.

In the above dialogue the students were basically asked to respond to the usefulness of viewing fractions as composite units of unit fractions. In reacting to this notion Renee targets a practice that is commonly used among students of all ages, that is the tendency to extend their knowledge of whole numbers to the rational number domain. This persistence in drawing a connection, albeit incorrect, with something already familiar fully supports the contention that we must expose our students to the genuine connections that exist between these two domains. This natural connection can be revealed through an initial and continued focus on the unit concept.

Lesson 5 marked the end of the group lessons in the teaching experiment. As indicated by the presentation of each lesson, neither operations with fractions, nor properties of fractions were ever discussed in the five lessons. Since fractions had not yet been covered in the students' regular math class, any knowledge of fraction operations was previous knowledge that had been attained prior to the beginning of the fall semester of 1993.

The final contact with the students was through an individual teaching interview. The objective of these interviews was to ascertain if and to what extent the students would connect the concept of unit to operations with rational numbers. If this connection was not initially made by the student, the researcher attempted to advance the knowledge of the students through the use of similar whole number problems or direct questioning. Responses obtained during the interviews provided much of the data for answering the research questions and therefore are presented throughout the discussion in the next chapter.

CHAPTER FIVE

ANALYSIS OF THE RESEARCH QUESTIONS

While chapter 4 presented the results of each lesson individually, this chapter will attempt to provide a more comprehensive analysis in response to the research questions posed in the earlier chapters. My intent is to provide detailed descriptions of the students' work as it relates to each of the research questions. These descriptions will contain my analysis which will be further expanded in the next chapter.

The research questions can be categorized as those dealing with (a) unit formation, (b) obstacles in unit understanding, and (c) extension of the unit concept. These categories provide the headings for the sections of this chapter. Each section will address one or more of the following research questions.

1. Do preservice elementary teachers exhibit informal knowledge regarding unit formation?

2. Are they aware of their own construction of units?

3. What cognitive obstacles are encountered in the process of understanding the unit concept?

4. How will an awareness of the informal nature of the unit concept affect their problem solving performance on whole number addition and subtraction?

5. Will knowledge of the role of the unit concept in the whole number domain facilitate learning of concepts in the domain of rational numbers?

Unit Formation

Do preservice elementary teachers exhibit informal knowledge regarding unit formation?

While all of the students demonstrated the use of various units throughout the teaching experiment, this question may best be answered by the results of lesson 1. Prior to lesson 1 there was no mention of the word unit. Students were given counting tasks with the instructions that the researcher was interested in watching them count. Every one of the students used different unit formations in their counting processes. Further support of the existence of informal knowledge of units was provided by the students' journals. After lesson 1 the students were asked to define or describe a unit and then to think of nonmathematical situations in which the concept of unit was used. Some of their descriptions reveal a firm understanding of the basic notion of units. Mary Gail writes, "A unit is a member of something or a group of things. It is used in all aspects of our lives and promotes organization."

Other descriptions reveal the beginnings of more complex properties. Ann writes, " Unit - a particular part, group, or set of like or similar things. Units go in levels. For example, 2 [sic] units can be greater or less than each other."

Some of the examples given of situations in which units were used that demonstrated conceptual knowledge of unit formation included make-up kits, clothes drawers, photography packages, money drawers in a bank, and measures in music.

Are they aware of their own construction of units?

This question must be answered on an individual basis. It was apparent that Renee was dependent on unit formation by grouping. In every task she formed some type of unit. Many times these units were constructed by means of a visual process. For example, task 2 of lesson 2 involved grouping two packs of gum containing five sticks with two packs of gum containing 18 sticks, Renee actually drew out 46 sticks of gum and visually found units of two as indicated by vertical lines. The discussion of each task revealed that Renee was aware of her construction of units, she stated, "I grouped. I group everything....It's easier for me."

Mary Gail seemed to be in direct contrast to Renee. Although she used a greater variety of units in her counting activities than the other students (units of 1, 2, 3, 5, and 100) she was not aware of this unitizing.

In discussion of the counting activities the following dialogue took place.

Tena: When y'all counted yesterday, were you aware, okay, I'm going to group by twos or I'm going to group by fives?

Mary Gail: It just came naturally.

Renee: Naturally.

Further support that Mary Gail is unaware of her construction of units is found in the dialogue following lesson 2. The students were asked if they saw any similarities between lessons 1 and 2.

Mary Gail: Um, not really. I saw today's task as more of a grouping project, whereas Tuesday I saw it as seeing how many one units there were. Today was how many groups of two or six there are, but today there was a lot more grouping for me. I know most of you grouped a lot Tuesday.

Tena: But you grouped, like yesterday, your group of 100. But you really weren't thinking...

Mary Gail: No. I didn't mentally think about it.

Renee: ...in all the problems I grouped, but I grouped everything on Tuesday. The idea of grouping was there both days for me.

The above dialogue reveals the extreme views of the students. Renee's obvious awareness of her construction of units and Mary Gail's denial of unitizing. The other students were more in the middle. Carolyn and Lisa unitized first and then counted which might indicate an awareness of unitizing. Ann just began to count but during the process of counting she formed units of various sizes. This would seem to indicate that her unitizing was more on a subconscious level like Mary Gail's method. For all of the students, with the possible exception of Renee, conscious knowledge of the construction of units in the counting activities seemed to be absent. The unitizing that was done was merely perceived as a means of keeping track as indicated by the following dialogue.

Renee: It's easier like if something interrupts you. It's easier to go back and check and see where you were. Whereas if you were just counting by singles, you'd have to start all the way over.

Mary Gail: That's where making separate piles come into it. Like when I counted off 100 cubes, I pushed that to one side so I knew, well that was 100. So I know even if I mess up, I don't have to count that 100 again.

The degree to which the students were initially aware of their unit construction can not be determined; however, it was obvious that the students became aware of various unit constructions through the discussions that followed the activities. When asked for alternate ways to complete the tasks all students were able to describe methods involving a different unit construction than those of their original method.

Obstacles in Unit Understanding

The intention of this section is to report some of the problems encountered by the students as they attempted to understand and generalize the unit concept.

What cognitive obstacles are encountered in the process of understanding the unit concept?

To help develop an understanding of the unit concept the students were given experiences in a variety of situations in which various units could be constructed and then actually construct units without direct instruction. In essence they had to "do" unitizing. There seemed to be two main obstacles that blocked the initial reaction of unitizing: Algorithm dominance and mathematical perception.

One of the biggest obstacles faced by the students while trying to conceptualize and extend the unit concept can be described as algorithm dominance. Since all of the students had many previous years of mathematics they had been exposed to numerous rules and algorithms. When presented with a new situation their first reaction was to recall the correct process that had been stored in their memories. When these processes could not be immediately recalled, they were lost because they had no back-up system for determining an answer. This lack of conceptual knowledge in our preservice teachers is certainly not a new discovery (Graeber et al., 1989) but is no less disturbing when it surfaces.

The dominance of previously learned methods first surfaced in lesson 2. The students were given worded problems that could be solved by unitizing or by using the more traditional approach which involved multiplication and division. After all students used the traditional approach the following dialogue occurred:

Tena: OK, why do you think, like yesterday when everyone counted, everybody grouped. But when I gave you problems like this, almost everyone of you switched back to units of one, where you kind of just dumped everything out and pulled.

Lisa: Easier. When you're counting hands-on there is more chance of a mistake because your mind wanders, things happen, so if you group everything it's easier to keep track of how many groups you have. In a math problem, when you're doing it on paper, I think it's easier to figure it out in ones because that's the way we've always done it. I mean that's the way I learned to do it.

The above discussion suggests that the method for completing the tasks of lesson 2 was based on habit. This is further supported by the fact that all students were able to provide a correct solution using units other than one when asked for an alternate approach. So while the unit concept was conceptualized well enough at this point to provide a correct solution to these tasks, the more traditional approach was chosen because "that's the way we've always done it" (Lisa).

This pattern of algorithm dominance continued throughout the teaching experiment but was especially apparent in the teaching interviews. The following comments are taken out of context but reveal the algorithm dominance that surfaced when students were asked about rational numbers:

Renee: I think you are programmed. You always know that you have to convert whenever you don't have the same thing. Right up there with the rule that says you have to leave everything in a line. All those rules just kind of stick out.

Lisa: Programmed in my mind. The first thing you always did was get a common denominator, then you worked it out, then you changed it to a mixed number.

Lisa: All I remember the teacher saying is that you could not say flip. You didn't flip the numbers, you had to take the reciprocal of the numbers. I never really knew what the difference was between the reciprocal and flipping.

Mary Gail: Cause we were told that. I mean I went to school a lot earlier than most of the students here and we just had things hammered into our head.

Carolyn: I didn't know how or why you had to do it, I just did it...That's my basic idea of math. You did it this way because that's the way it is supposed to be done and you don't ask questions.

The other main obstacle, which might be considered an activator for algorithm dominance, was mathematical perception. By this I mean how mathematical the student perceived the problem to be. If the problem was perceived as a math problem the unit structure was usually limited to units of one and algorithm dominance was present. The more realistically the problem was perceived, the more flexibility demonstrated in the unit structure. A good example of the obstacle of mathematical perception can be seen in the prom problem of lesson 4. Recall that in this task the students were asked to plan the number and size of the tables needed to accommodate the various groups attending the prom. The first reaction by the students was consistent with earlier reactions to problems perceived as math problems, they calculated the total number of people so they could divide. Through the discussion the problem became more realistic to the students and this original approach was abandoned. Even upon finding a workable solution to this problem, the students made additional changes based on social aspects. The following dialogue occurred after the students realized that their current solution would leave one couple out of the couples only section:

Renee: Now, you may have a fight with these two people that are left over unless you just threw them in.

Mary Gail: Well, let's say everybody understands...

Renee: But you are saying that this couple can't sit in the couples only...

Mary Gail: If we go back to that, not everyone is going to sit at the same time, so do you really need that many tables?

Renee: But every girl needs a place to hang her purse. She doesn't want someone else sitting there if she goes to dance.

Lisa: But make this an eight person table, leave this couple out and make this...

Renee: That would add one more table. You could make a two person table and then...

Lisa: Then everyone else would say why didn't we get our own table...

The above discussion provides evidence that the prom problem became more than just another routine math problem. The more realistic the problem became the more factors they needed to consider. It also illustrates a willingness to adjust the unit based on a social problem rather than a numerical one.

The flexibility of the unit that occurred once the prom problem became real was also visible in other tasks. In lesson 3 the students were asked to arrange eight chairs around two tables. Most all of the arrangements involved four people per table. During the discussion of the results of the task the realistic factor began to dominate.

Tena: Is there another way we could have done it?

Lisa: Five and three.

Lisa: All at one table.

Tena: OK. We could crowd all of them around one table and leave the other table empty.

Carolyn: Or put seven and the other person nobody likes at the other table [laughs].

Lisa: I was thinking of what looks best and two tables of four looks better to me.

Carolyn: If you just had people coming in randomly of course they are going to sit by someone they know, so they might not work it out the way you want.

The above dialogue provides another example that the unit structure used within a problem is dependent upon the mathematical perception of the problem. When the problems were first approached they were treated as routine math problems and were dominated by units of one solutions. Once the problems were discussed in a realistic text the solutions involved units of various sizes. This would suggest that one of the main barriers to understanding and applying the unit concept could be eliminated if math problems were presented in a more natural and realistic manner. Carolyn expressed the need for a more realistic connection between school math and everyday life early in the teaching experiment:

Carolyn: My mother, I don't want to blame this on my mother so much but I mean if she had sent me to the store and said, OK, we need this, this, and this, and you have to multiply, and divide this, and we only have this much money but we need this. You know. Or what is the better buy 24 oz or the 16 oz. I think, you know, you would have connected it all and said Oh yeah this all works.

Tena: So, in other words, making, taking more realistic situations?

Carolyn: Right.

It is very interesting to note that Carolyn put the burden of a realistic connection on her mother rather than her teachers. This might be an indication of an even bigger obstacle that would justify algorithm dominance and mathematical perception: Teachers teach school math and real mathematical applications occur only in non-school settings.

Extension of the Unit Concept

How will an awareness of the informal nature of the unit concept affect their problem solving performance on whole number addition and subtraction?

This question was partially answered in the previous discussion of obstacles the students encountered which seemed to prevent the initial use of the unit concept. While most of the whole number problems were solved using the traditional unit of one approach, the second effort solutions revealed that the students were capable of applying various unit formations to the whole number problems. A nice example was provided by Carolyn's second attempt at the Bubble gum problem. Recall that this problem involved two packs of gum containing five sticks and two packs of gum containing eighteen sticks. The students were basically asked to determine how many groups of two. Carolyn's second attempt reveals a nice understanding of the unit concept, she explains,

> I was just sitting here and it occurred to me that you could just take the five sticks and add the 18 to it and then you would get the 23 because they're already broken into twos, two packs of five and two packs of 18.

Carolyn's solution demonstrates a maturing of the unit of one approach. The 2(5 stick packs) and the 2(18 stick packs) were reunitized as 5(2 unit)s and 18(2 unit)s which would yield 23(2 unit)s. This solution also supports the previous contention of the role of realism in mathematical perception. Carolyn states, "I guess if we had taken these out of the bag and really looked at them it would have clicked." Again the notion that as the problem became real the choice of the unit was altered.

Formation of various units was not always limited to second effort solutions. In the Donut problem of lesson 2, many of the initial solutions demonstrated an extension of the unit concept to whole number operations. Ann described her first solution, "I did pretty much what they did. I just went 12 donuts divided by 3 equals four snack packs for each dozen times 4 dozen equals 16 snack packs." When asked about the focus of her unit structure Ann replied, "I did the 12. In a dozen there's four three-units times 4 dozen-units."

Further evidence to suggest that the unit concept was impacting whole number operations was provided in the discussion at the conclusion of lesson 4. The following discussion was generated by asking the students to add 12 inches plus three feet.

Renee: Well my first impression is to say three feet, 12 inches.

Mary Gail: Four feet.

Tena: OK. If you said four feet what are you changing?

Mary Gail: I converted 12 inches to one foot. You're looking at 12 single units as...

Tena: A one foot unit?

Mary Gail: Right.

Tena: Could you do it another way?

Lisa: You could convert the feet to inches.

Tena: So what does that give you?

Lisa: Four 12 inch units.

Ann: Or you could go 12 one inch units and 36 one inch units

Tena: OK. What does that give you?

Ann: 48 one inch units.

The above discussion revealed that the students were becoming more comfortable with conceptualizing a variety of unit approaches. This was extended to a basic whole number addition by the next question.

Tena: What if I just give you two whole numbers, six plus four. If I ask you to find something about those so that you are not adding six one things plus four one things...

Mary Gail: They are both divisible by two so three two-units and two two-units, gives you five two-units.

Tena: Which converts to what if I go back to ones at that point?

Mary Gail: 10 units.

Will knowledge of the role of the unit concept in the whole number domain facilitate learning of concepts in the domain of rational numbers?

The discussion of rational number operations was reserved for the teaching interviews. The researcher was very careful not to address operations with rational numbers until this point. This was deemed necessary if the reactions to the interview questions concerning rational numbers were to be based on conceptual understanding and not on memorized algorithms. It was felt that the students would not be able to provide an intuitive reaction if the traditional algorithms had recently been reviewed. For this reason the students were taken from their regular mathematics class before the review of rational numbers took place.

While the format of each interview varied depending on individual responses, there were two main areas that were examined with each student. The first area of interest was whether or not the students would notice and/or use the unit concept in addition of fractions. If this connection was not naturally made by the individual then their reaction to the suggestion of the unit concept was examined. The use of the unit concept in addition of fractions was further examined by asking the students to respond to a child who added numerators and denominators (e.g., $2/3 + 1/2 = 3/5$).

The second area of interest concerned division of rational numbers. Once the unit concept was examined in the operation of addition, could the student extend this to the operation of division (e.g., $9/12 \div 3/12$). Here the emphasis was on whether or not the students would connect the unit concept to this operation.

Each interview began by asking the student to add two fractions with common denominators (e.g., $4/5 + 3/5$). The students were provided with paper and pencil and were given the option of writing the problem or working it mentally. After their response the students were asked to justify their answer.

All of the students used the traditional algorithm in their justification except Lisa. She seemed to immediately recognize the unit structure as illustrated by the following dialogue.

Tena: If I asked you, four-fifths plus three-fifths?

Lisa: Seven-fifths.

Tena: OK. How do you know?

Lisa: Because four plus three is seven.

Tena: OK. When you thought about that as four plus three are you looking at that as, numerator plus numerator?

Lisa: Uh-huh.

Tena: OK.

Lisa: And four five units plus three five units...

Tena: OK...

Lisa: is seven five units.

Lisa's use of the unit concept in her justification came without hesitation. While her understanding of the unit concept seemed intact, her use of five unit rather than one-fifth unit was further examined.

Tena: What if I asked you two-sevenths plus three-sevenths?

Lisa: Two-sevenths plus three-sevenths? Five-sevenths.

Tena: OK...because...

Lisa: Same thing, two seven units plus three seven units.

Tena: OK, when you say seven units, if I asked you three seven units plus four seven units?

Lisa: One.

Tena: Alright...

Lisa: Because four plus three is seven and then just automatically I would change that to one.

Tena: OK, because you are looking at that as seven over seven?

Lisa: Right.

Even though Lisa's terminology is a little weak she is obviously conceptualizing the five unit as one-fifth and the seven unit as one-seventh. In order to see if Lisa would correct her unit terminology the discussion continued.

Tena: ...Is there a difference in what you are saying if I had this [writes 3(5 units) + 2(5 units) on the paper]. Is that the same? Are you looking at that like this [writes 3/5 units + 2/5 units]?

Lisa: No, it is not the same. Well, it depends....No, it would be more three units of five.

Tena: Alright...

Lisa: and three five units. Because three five units I look at as one unit, two units, three units of five or 15 small units [uses her hands to make three imaginary groups on the table].

Tena: OK.

Lisa: But with three, What did I say before? Three?

Tena: you said three...

Lisa: Three of five units. I don't know how to say it.

Tena: How could we make a distinction?

Lisa: Like if we had [she begins drawing five squares on the paper and marks through three of them] five units and three-fifths would be three of the five units.

Tena: OK. So how big is each one of these units [points to one of the squares]?

Lisa: One-fifth.

Tena: OK, so that would be like this is three...[points to 3/5]

Lisa: one-fifths.

Tena: Units?

Lisa: uh-huh.

Tena: and this would be [points to 2/5] two...

Lisa: one-fifth units.

As a result of this dialogue, Lisa seemed to have corrected her terminology but at the same time she has indicated that this correction may not have been necessary. My initial concern with her response using five units rather than one-fifth units was that Lisa was disregarding the unit fraction as the unit. In her explanation of three-fifths given above this is clearly not the case. It is interesting to note that Lisa's drawing (⊠⊠⊠□□) first appears to be a ratio diagram for three-fifths rather than a part/whole diagram but her explanation suggests a combination of these approaches. Instead of dividing one square into five parts and shading three of them, she draws five squares and shades three of the five squares. This would seem to be a typical ratio diagram. However, when asked about the size of one square she immediately replies "one-fifth" which suggests that she considers the five squares to be a whole so one square is part of that whole. The uniqueness of Lisa's approach is supported by my previous 10 years of experience with preservice elementary teachers. When asked to illustrate a fraction such as three- fifths the most common responses include a ratio diagram (⊠⊠⊠□□) or an area model which illustrates the part/whole relationship (▨▨▨□□). In the ratio diagram each shaded square is considered to be a whole whereas in the area model each shaded square is considered to be one-fifth of the whole. Lisa's reference to one of the squares in her diagram as one- fifth suggests a combination of the two typical approaches. Her unique approach supports the notion that the unit fraction one-fifth is being conceptualized as the iterated unit to obtain three fifths. So while her use of seven units instead of one-seventh units initially signaled a misunderstanding to the researcher, further probing suggested that the conflation in terminology indicated a crossfertilization of whole number and rational number concepts.

While Lisa's initial responses seemed to reveal such a sound understanding of the unit concept, this thinking appeared to be abandoned when asked to add fractions with unlike denominators. Her first explanation of three-halves plus one-fourth consisted of the traditional algorithm. However, when asked to respond to a child who writes 2/3 + 1/5 = 3/8 she again reveals a mature understanding of units.

Lisa: He's adding these together [points to numerators and denominators] and he shouldn't be. I would probably go back to this [refers to her diagram] to explain it, saying this is not a unit [circles the two in the numerator]. This is the total unit [circles the three in the denominator], and this is how much of a unit that you actually have.

Although Lisa did not draw a diagram for two- thirds the above descriptions suggest that it would follow the same pattern as the diagram of three- fifths. She would draw three squares which she refers to as "the total unit" (Lisa) and then she would shade two of them, corresponding to "how much of a unit that you actually have" (Lisa). When asked about the size of each square, she would respond with one-third which again would indicate the size of the iterated unit.

This unique approach of Lisa's suggests a strong understanding of composite units. To Lisa the denominator represents the "total unit" (Lisa). For two-thirds this means 1(3-unit) but the 3-unit is composed of 3(1/3 unit)s which suggests a structure of 1(3(1/3 unit)s-unit). The numerator indicated the number of these nested units that you actually have. So two-thirds is 2(1/3 unit)s.

While this level of understanding was not demonstrated by all of the students, the others were able to identify the unit structure in addition of fractions with a little probing.

Tena: If I go back to the first one, four-fifths plus three-fifths, do you see any similarities between two balls plus three balls and four-fifths plus three-fifths?

Mary Gail: I'm really not using the denominator, I'm just using the top numbers as single units.

Tena: Why aren't you using the denominators?

Mary Gail: Because they are the same, so I know, when I give my answer. Like when you said three balls plus five balls or whatever, I know that it is three over one plus five over one, they have the same denominator.

An interesting observation in the dialogue above is that while others tend to add fractions the same as whole numbers, Mary Gail is adding whole numbers like she would add fractions, "three over one plus five over one."

The interview with Ann also led to the connection between whole number addition and rational number addition as indicated by the following dialogue.

Tena: I'm going to start with just asking you to add two fractions for me, four-fifths plus three- fifths.

Ann: Seven-fifths.

Tena: OK. How do you know?

Ann: Cause there is a common denominator, five, so you just add the top ones.

Tena: OK.

Ann: You'll come up with seven-fifths.

Tena: OK. What if you had two-sevenths plus three-sevenths?

Ann: Five-sevenths, same thing.

Tena: What if I asked you five inches plus six inches?

Ann: Eleven inches.

Tena: What is 14 balls plus two balls?

Ann: Sixteen balls.

Tena: What is three-twelfths plus five-twelfths?

Ann: Eight-twelfths.

Tena: Do you see any similarities between those problems?

Ann: You are adding the same thing in all of them, I mean a common unit.

Tena: OK, so when you see the three-twelfths plus five-twelfths, what's common?

Ann: Twelfths.

After the discussion of addition of rationals the students were given the problem nine-twelfths divided by 3/12 and asked to respond. While the initial response of some of the students was correct their inability to recall the traditional algorithm left them insecure. This can be seen in the following dialogue with Renee.

Tena: Alright, what if we switch to division [writes 9/12 ÷ 3/12]?

Renee: I haven't done this in so long. Do you change...? 12 divided by 12 would be a one, so would it be three?

Tena: OK. Now think out loud.

Renee: I'm trying to remember actually when you divide the fractions if you leave the denominator the same, or if you divide outright. So either that is going to be three or one-fourth.

Tena: OK. So you have a choice?

Renee: We'll go with three.

Renee's instinct is to give a quotient of three but she can not support this because she has forgotten the algorithm. The dialogue with Lisa also seems to indicate a struggle between the algorithm and the instinctive approach.

Tena: OK. Let's look at [writes 9/12 ÷ 3/12].

Lisa: [hesitates] Three-twelfths.

Tena: OK. How did you get that?

Lisa: I have no idea. I haven't divided fractions in a really long time.

Tena: OK, was that an algorithm or did that just seem to make sense to you?

Lisa: It seemed to make sense because you added these two together [refers to the numerators in previous addition problems]. Wait, don't you take the reciprocal and multiply?

Tena: Alright, that is the algorithm.

At this point Lisa works the problem using the algorithm and gets three. The discussion continues.

Tena: So were you right?

Lisa: Right.

Interesting to note that although Lisa's first response was "three-twelfths" she seems to consider this the same as the three which was obtained from the algorithm. A possible explanation could be that to Lisa the twelfths is merely a label. The dialogue continued.

Tena: But now in your mind, did you do that? [refers to the algorithm]

Lisa: No. I just divided the top numbers. Just one of the short cuts.

Tena: OK. What if I had given you [writes 27/35 ÷ 9/35]?

Lisa: Same thing but I'm not sure if it is right or not. See this is the way I would do it in an algebra problem [refers to the algorithm], but if I was in a grocery store I would probably just divide the nine into the 27 to get the three.

The above dialogue lends further support to the obstacle of mathematical perception discussed earlier in this chapter. Lisa's instinctive approach is good enough for the grocery store but not for math class.
The two approaches are further examined.

Tena: Which one makes more sense to you?

Lisa: Just dividing.

Tena: Just your instinct?

Lisa: Uh huh.

Tena: Would you still want to do that if I put different denominators [writes $27/15 \div 9/12$]?

Lisa: No. I would go back [points to the algorithm].

Tena: You would go back to the algorithm?

Lisa: Right. If I wanted to I could find a common denominator but that would take more time. It would be quicker to just take the reciprocal.

The above dialogue indicates that Lisa has developed an alternative method for division of fractions. Her new approach which is based on instinct consists of getting the fractions to a common denominator and then dividing the numerators. Although Lisa states that this approach makes more sense to her she also states that she would probably use the traditional algorithm if the original problem did not have common denominators because it is faster. It is fascinating to note that this new approach to division of fractions was realized by all of the students in the study. Ann discovered the new approach while trying to explain the algorithm as indicated by the following dialogue.

Tena: Why do you invert and multiply?

Ann: You want to see how many times this will go into this, three-twelfths into nine-twelfths.

Tena: Alright.

Ann: and, let's see, you know that three- twelfths is bigger than, I mean it is smaller than nine-twelfths and, well, gosh, can you just say nine divided by three is three? I mean it looks like it.

Tena: What do you think about that?

Ann: That's the same answer we got. As long as you have the common denominator can you do that? I don't know.

Ann seemed excited by her new discovery and checked her hypothesis by using the traditional algorithm. The discussion resumed.

Tena: So, would that work every time?

Ann: I assume it would. I've never seen that before.

Tena: What if you had [writes 27/363 ÷ 9/363 on the paper], would your new strategy work?

Ann: Uh huh. I guess, cause you have 27 of one thing and nine of the same thing so, three.

Ann's explanation suggested that she had become sensitive to the role of the unit concept in division of fractions. At this point Ann was given a problem with different denominators. Just like Lisa, Ann concluded that the new approach would work, but that the traditional algorithm would be quicker.

The construction of a new approach to division of fractions realized by all of the students was very exciting. The degree to which this discovery can be attributed to the knowledge of the unit concept attained in this study can not be determined. It can merely be said that while the students had worked with fractions for at least 10 years prior to the study this conceptualization of division was not realized until the end of the teaching experiment. This provides strong evidence that knowledge of the unit concept seems to facilitate understanding of concepts in the domain of rational numbers.

Chapter 5 presented an analysis of the research findings in reference to the individual research questions of the study. The comprehensive analysis as well as the implications for pedagogy and research will be discussed in chapter 6.

CHAPTER SIX

SUMMARY AND CONCLUSIONS

This chapter presents the conclusions ensuing from this study. The chapter begins by providing a general overview of the study based on a synopsis of the first three chapters. Then the general conclusions will be discussed. These conclusions were deduced from the specific results discussed in chapters 4 and 5. Finally, the pedagogical implications and the implications for future research are addressed.

Summary

As an educator of prospective elementary teachers I am continually amazed at their lack of understanding of rational numbers. The thought of my own children (or anyone else's for that matter) being guided by someone with so little conceptual understanding was a disturbing. How could my class make a difference? What kinds of experiences should I provide to better help them construct knowledge of the rational number domain?

My readings in graduate school directed me to the concept of unit. I was excited by the possibilities and curious as to why this was new to me. I had been a student of mathematics for many years but I was never aware of the concept of unit. The more I read the more I was convinced that the unit concept was powerful. It all seemed so clear to me. This was the key to unlock those minds closed by all the memorization of mathematics. I thought the concept of unit had great promise and research agreed, but what about my students? How would they react?

To address these issues, a three week teaching experiment was devised to provide situations in which the students could explore the concept of unit. The teaching experiment consisted of five lessons. Each lesson was designed to illustrate various aspects of the unit concept including unitizing, reunitizing, and norming. The issue of the unit concept as a connector between the whole number and rational number domains was examined through individual interviews that were conducted at the end of the five lessons. These lessons, as well as the individual interviews, were videorecorded and transcribed. In addition, a journal consisting of reflections, spontaneous observations, questions, ideas, etc., was kept. Further information was gathered through students' task sheets, homework, and essays.

The lessons of the teaching experiment were exciting. I loved watching their reactions when they became aware of the unitizing that they performed spontaneously. I was fascinated by their individual progress, not only with the unit concept, but with communication. Those who had begun the study with hesitance were becoming verbal and revealing some of their ideas and misgivings. Those students whose grade would suggest a below average ability were responding to the tasks and generating solutions quicker than those whose grades would suggest an above average ability. The students seemed to be equalized in their abilities, perhaps because this concept was not school-taught.

After the enjoyment of the teaching episodes, the task of analyzing the data was monumental. After many, many hours of reading, things began to fall into place. The process used to analyze the data was based on the methods described by Bogdan and Biklen (1982) which consisted of the development and implementation of a coding system and then sorting the data based on these codes. Analysis was conducted within and between the sorted data.

Conclusions

The results of the data analysis revealed much about the conceptualization of the unit. The intuitive notion of unit suggested by von Glasersfeld (1981) and discussed in chapter 1 was supported by the findings of this study. The students' performances on the individual tasks clearly showed the formation of various units. The suggestion that this formation was intuitive is supported by their individual comments when asked if they were aware of the various units they were forming: "It just came naturally" (Mary Gail), "Naturally" (Renee), "No, I didn't mentally think about it" (Mary Gail).

Further support of the intuitive nature of forming units surfaced when the students were asked to name other situations in which units were formed. The students began to describe numerous nonmathematical examples of units such as measures in music, photo packages, organization in drawers, closets, and cupboards, etc. This overwhelming display of unit suggestions revealed that the students were becoming sensitive to the notion that their previous attention to organization could be interpreted as the formation of units.

While the intuitive nature of unit formation became evident, obstacles to the understanding and implementation of the unit concept also became apparent. It was noticed that unit formation was somewhat hampered by the students' previous knowledge of school mathematics. The extent of unit involvement in a given situation appeared to be dependent upon overcoming the obstacles of algorithm dominance and mathematical perception. As described in chapter 4, these obstacles refer to the interference caused by the conflict between old or previous knowledge and the new knowledge. While the students were able to generate second solutions to many of the activities involving various unit structures, the original responses were usually dependent upon previous algorithms or teacher-taught methods. This tendency to suppress an instinct in favor of a memorized approach illustrates the notion of algorithm dominance.

It was also clearly revealed that the unit structure chosen for the solution process was dependent upon how mathematical the problem was perceived to be. As the problems became more realistic to the students, the more flexible their unit structure became. This notion that the approach to school mathematics differs from that of street math has been well documented (Brown, Collins, and Duguid, 1989; Collins, Brown, and Newman, 1989; Lave, 1988). While Lave (1988) would argue that all of the tasks in this study were considered school tasks because of the laboratory type setting, it was interesting to note that as the discussion about the tasks progressed the mathematical surroundings seemed to be forgotten and the realistic features of the problem began to surface. This was especially apparent in the Prom problem of lesson 4. In this problem the students had to establish the size and number of tables for the school prom. The task was complicated by the fact that groups of various sizes were attending the prom and they wanted to sit together. Numerous attempts at solutions were discarded because of social factors. The confrontation between the approaches to school math and those used in street math describes the obstacle of mathematical perception. Although the students had to overcome the obstacles described above, they were still able to find connections to the concept of unit.

To many students, mathematics is perceived as a collection of isolated facts with different rules for different occasions. Their attempt to learn mathematics is often limited by their ability to memorize and then recall these rules. The students observed in this teaching experiment were no exception. During the discussions of the lessons many of the students used the word *programmed* to describe their experiences with traditional school mathematics.

The current national reforms in mathematics education are attempting to dispel this rule dependence by stressing the importance of teaching for connections in mathematics. This is perhaps most apparent in the NCTM's <u>Curriculum and Evaluation Standards for School Mathematics</u> (1989) which states:

> The fourth curriculum standard at each level is titled Mathematical Connections. This label emphasizes our belief that although it is often necessary to teach specific concepts and procedures, mathematics must be approached as a

> whole. Concepts, procedures, and intellectual processes are interrelated. In a significant sense, "the whole is greater than the sum of its parts." Thus, the curriculum should include deliberate attempts, through specific instructional activities, to connect ideas and procedures both among different mathematical topics and with other content areas. (p. 11)

The search for a connector between the ideas and procedures of the whole number and rational number domains was one of the main catalysts for this study. While attempts to connect the tasks to previous methods was apparent throughout the teaching experiment, it is exciting to note that there were apparent connections made between the whole and rational number domains. One of the most interesting was concerning division of rational numbers. During the individual interviews all of the students were able to generate an alternative to the traditional invert and multiply algorithm when given a problem like nine-twelfths divided by three-twelfths. Their new method involved obtaining common denominators and then just dividing the numerators. While this approach may not be new to the mathematics community, it was new to these students. All of them indicated that they were taught to divide fractions by using the traditional invert and multiply algorithm.

While the notions of units developed over the study were an instrumental aspect of the interview structure, attributing the alternative approach to division of rationals to the unit concept is still tenuous. Future research is needed to make this attribution stronger.

Implications for Teaching and for Research

Although the results of this study do not provide an overall solution to the problem of developing rational number concepts, they do provide evidence to warrant some suggestions regarding pedagogy and future research.

The review of the literature in chapter 2, while not exhaustive, was enough to reveal that the current approach to teaching rational number concepts is not effective. This study provides evidence that curriculum development should provide experiences with problem situations emphasizing mathematics of quantity (Behr, Harel, Post, and Lesh, 1992). Through this emphasis the basic principles of the unitizing concept would be internalized before exposure to rational numbers thereby providing a link between the understandings used in whole number operations and those used in rational number operations. This curriculum development would involve exposing students to whole number situations in which different units were present. Take for example, five balls plus two boxes of balls. Here the student can not obtain the answer by simply removing the numerals from context and adding which is presently the case in many whole number addition problems. By continuing this focus on the unit as a quantity, consisting of size and number, the conceptualization of 2/3 + 4/5 as 2(1/3 unit)s + 4(1/5 unit)s would be supported. In the latter notation the student can see the resemblance to whole number addition as well as the need to obtain units of the same size in order to perform the operation.

Results of this study revealed that the artificial nature of mathematics problems sometimes hinders the implementation of the unit concept. However, the data revealed that the students' discussions of the problems generated a more realistic interpretation, thereby providing a more intuitively grounded approach to the problems. This would suggest that one attempt to make current textbook problems more realistic might simply entail a classroom environment in which groups of students could openly discuss and interpret the situations presented in the problems.

Regarding implications for future research, one of the barriers to the implementation of the unit concept was the dependence on previously memorized algorithms. It would be interesting to observe what, if any, obstacles would surface if the unit concept was emphasized before the algorithms. This was not possible in the present study since the students involved were at the college level. A study involving young children who had no prior knowledge of rational number operations might be beneficial in determining the power of the unit concept as a connector between the whole and rational number domains.

While previous research has addressed the notion of units and suggested its importance in numerical development (e.g., Saenz-Ludlow, 1994; Behr et al., 1993; Steffee and Spangler, 1993), these conclusions have not been expanded to the effect the unit concept can have on classroom teaching. In this regard this study represents a ground breaking effort. It is a first attempt to take basic cognitive research and investigate its applicability in a situation more closely modeling that of a normal classroom.

In communications with my students I am often asked how a certain concept will be addressed at various grade levels. The impact of the unit concept on the elementary school classroom is an important question but not addressed here. What is the current development of the unit concept in classrooms and textbooks? Are students being exposed to situations involving units other than one? Unless our current teachers are reading the research journals how will they become aware of the importance of the unit concept? Perhaps the fastest way to update our current approach to units is through our preservice teachers. If our future teachers become aware of the importance of the concept of unit and the connections that are possible, our children will be reached.

A traditional quantitative study might lend further support to the power of the unit concept. A study could compare the performances and understanding of rational number concepts between a group of students exposed to the concept of unit with a group of students taught to use the traditional algorithms.

Epilogue

The study of mathematics is often met with fear and anxiety. Students of all ages cling to formulas and mnemonic devices in hopes of remembering the enormous collection of facts and rules that are seemingly unconnected. Perhaps this fear is most present in the minds of those individuals considering a career in teaching. The thought of being responsible for guiding the mathematical development of a room full of children can be overwhelming even for those few who feel confident and prepared. For the most part, when asked about teaching mathematics, our preservice elementary teachers indicate a lack of confidence stemming from a lack of conceptual understanding. This dissertation was undertaken in hopes of identifying a way to promote understanding by finding a connection among all those facts and rules. The underlying theory was as follows: Is the formation of units an intuitive process? If so, can an awareness of this intuitive knowledge be extended to provide a foundation for linking the realm of understanding between the whole and rational numbers.

The analysis of the findings of this study are promising. All students formed various units even before this concept was discussed. This supports the contention that unit formation is indeed a natural or instinctive occurrence. Problems involving whole numbers were re-worked to demonstrate solutions of various unit structures. There was also clear evidence to suggest that the concept of unit could provide a link between the whole number and rational number domains. This link might be made stronger with a more direct instructional treatment that calls attention to the fact that one-third in five-thirds, for example, behaves like inches in five inches, bats in five bats, balls in five balls, etc. This researcher can envision an even broader scope of the unit concept encompassing notions of algebra like combining like terms, laws of exponents, solving exponential equations, adding rational expressions, etc. Imagine the reaction to a mathematics curriculum that is built around knowledge that we all possess. A curriculum that demonstrates connections from the counting activities of kindergarten all the way through to the manipulations of algebra. The concept of unit can make a difference in the conceptual understandings of our students. I witnessed this in a short period of time with a group of students who were preconditioned by the traditional procedures of the traditional school curriculum. What would the possibilities be if the period had been longer, the students less programmed, the approach more direct? Perhaps through future study of the concept of unit we may help quell the fear of fractions for future generations of students.

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APPENDICES

Appendix A: INVENTORY

Initial Problems

The four problems below will be given to the students on separate sheets of paper. The students will be asked to show all of their work in solving these problems.

Problem 1. Given 5/6 + 1/2 + 2/3

a) Perform the indicated operation - SHOW ALL WORK

b) Please provide a written explanation for the process you used in part a).

Please show all of your work to the following problems:

Problem 2. Billy plans to outline his driveway with miniature American flags for the Fourth of July. He has three boxes of 10 flags left from last year. If he buys two boxes of 20 flags and borrows 25 flags from his neighbor, how many flags will he have to display?

Problem 3. Mary is having a small party and is planning to serve tacos from a local fast food chain. She orders three 6-packs of soft tacos and one 10-pack. If every guest is to receive two tacos, how many people can she invite?

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Problem 4. After the class pizza party there was 3/4 of a 16-slice pepperoni pizza and 1 1/2 of a 12-slice sausage pizza left over. Mrs. Adams decided to freeze the remaining pizza by filling freezer bags with three slices of pizza to a bag for her children to have for after school snacks. How many freezer bags are needed?

Additional Problems

The problems below will be given to the students after they have worked the above problems. The six problems below will be placed on separate sheets of paper. The students will be asked to sort all ten problems into categories and then they will be asked to explain their criteria for sorting.

Problem 1. $(3/4 - 2/5) \div 1/2$

Problem 2. Mrs. Martin needs more books for the music class. While cleaning out the store room, the school janitor finds four boxes of old songbooks. One of the boxes contains 15 books, two of the boxes contain 24 books each, and the last box has only 9 books. If Mrs. Martin assigns three students to a songbook, how many students will benefit from the janitor's discovery?

Problem 3. A local cafe orders lettuce in boxes with 12 heads in each box. While taking inventory after a busy lunch crowd, there remains 1/4 of one box and 2/3 of another box. The manager must decide if he has enough lettuce to service the evening crowd. He allows one head of lettuce for every four people. If the manager orders no more lettuce, how many people can he serve that evening?

Problem 4. The city has eight fire trucks. Each truck holds 42 gallons of gas. If three gallons of gas cost $3.27, how much does it cost to fill one truck?

Problem 5. Josh has three sacks with four marbles in each sack. Stephen gives him two sacks with six marbles in each sack. Josh is playing a game in which each player needs two marbles. How many people can play?

Problem 6. Jane's sticker book has four empty pages and three pages that are 1/2 full. If each page will hold eight stickers, how many more stickers can she add to her book?

Sorting Sheet

You are asked to sort the ten problems into categories. You should decide what problems should be grouped together and why. In the space below please indicate the problems that you grouped together and explain why you grouped these particular problems (for example: I grouped numbers -,-,and- because...).

Appendix B: LESSONS

Lesson One

<u>Task 1:</u>

a) Please take a bucket of unifix cubes from the table and count the cubes in that bucket. Indicate the number of cubes in the blank provided.

________cubes

b) Please describe the process you used to count the cubes.

<u>Task 2:</u>

a) Please count the number of faces that appear in the following picture.

☺ ☺ ☺ ☺ ☺ ☺ ☺ ☺ ☺

☺ ☺ ☺ ☺ ☺ ☺ ☺ ☺ ☺ ☺

☺ ☺ ☺ ☺ ☺ ☺ ☺ ☺ ☺ ☺ ☺

There are ________faces.

b) Please describe the process you used to count the faces.

<u>Task 3:</u>

a) Please take a bag of popsicle sticks from the table. The bag contains bundles of various numbers of sticks. Please count all of the sticks in the bag. You may remove the rubber bands if you like.

There are _________sticks.

b) Please describe the process you used to count the popsicle sticks.

Task 4:

a) Please refer to the poster on the table. Pretend that you are standing on top of a tall building and you are looking down at the crowd below. The dots on the poster correspond to the heads in the crowd. You need a good estimate of the number of people for the newspaper. How many people are in the crowd?

There are about _________people.

b) Please describe the process you used to approximate the number of people in the crowd.

Lesson Two

Task 1:

A local bakery has developed a new plan to improve the sale of donuts. Every morning donuts are boxed by the dozen in preparation for the morning crowd. By mid-day many of the boxes remain unsold. In an effort to promote the sale of donuts after 11:00 am, the manager has decided to sell donuts by the snack-pack. Any box left unsold after 11:00 am will be re-packaged as snack-packs and sold at a reduced price. If a snack-pack is to contain three donuts, how many snack-packs can be made from four boxes of unsold donuts? Remember, let my see your thoughts!

Task 2:

Susan is team mother for her son's baseball team. One of her duties is to provide bubble gum for the players. She has decided to buy the sugar-free gum that comes in regular packs of 5 sticks and family packs of 18 sticks. Susan allows two pieces of gum for each player per game and buys just enough gum with no extra pieces. If she buys two 5-stick packs and two 18-stick packs for the first game, how many players are on the team? (You will find items for this problem on the table).

Task 3:

Billy's mom is making party favors for his birthday. She plans to have sacks with candy and baseball cards to give each boy as they leave the party. She buys two packs with 15 cards per pack, three packs with 12 cards per pack, and one bargain pack with 54 cards. If she plans to put six cards in each bag, how many party favor sacks can she make?

Lesson Three

Task 1

Peter is trying to set up a display of soccer balls in his athletic shop. He has 24 balls to display on a table. The balls are in boxes to make them easier to arrange. For his first attempt, Peter tries 4 groups of 6 balls as shown below, but he doesn't like this arrangement. Please help

Peter set up the display by showing him 3 more ways the balls can be displayed.

You may use the cubes on the table to help with your arrangements. Please draw your final arrangements in the space below.

Peter's first try: □□□□□□ □□□□□□ □□□□□□ □□□□□□

Record your arrangements below.

Task 2:

Eight people must sit at the 2 tables below. Chairs are stacked in the corner of the room so each person must get a chair and take it to one of the tables. Please indicate 3 possible seating arrangements in the space below.

Task 3

After the faculty meeting the lecture hall was a mess. There were 2 tables with 6 chairs each, 3 tables with 4 chairs each, and 2 tables with 2 chairs each. Please straighten the room so that there are the same number of chairs at every table. Please draw your solution in the space below.

Lesson Four

Task 1

The junior class is busy planning for the Prom. The decorating committee is trying to decide on the number of tables it needs. There are 225 seniors but not all the seniors are planning to attend. According to the latest count their will be 56 couples, 15 groups of four people, eight groups of three people, and six people are coming alone. Please decide how many and what size tables are needed. Then explain how you would arrange the name tags.

Lesson Five

Task 1

Refer to the cardboard strips and the piece of tape on the table.

1. Use Unit A as a ruler to measure the tape (unit B).

1(B-unit) = ______(A-unit)

Explain what you did.

2. Use unit B as a ruler to measure unit A and unit C.

1(A-unit) = ________(B-unit)

1(C-unit) = _________(B-unit)

1(C-unit) = _________(A-unit)

Explain.

3. Use unit C as a ruler to measure unit A and unit B.

1(B-unit) = _________(C-unit)

1(A-unit) = _______(C-unit)

Explain what you did.

Task 2

Please use the same size fraction pieces to describe the following as composite units.

a) 2/3 and 5/6   b) 3/2 and 3/4   c) 4/5 and 1/2

VITA

Tena Long Golding was born in Jackson, Mississippi, on April 5, 1957, the daughter of William Wesley Long and Billie Jean Morehead Long. After graduating as Salutatorian from Horn Lake High School in Horn Lake, Mississippi, she attented Northwest Junior College in Senatobia, Mississippi on a Presidential scholarship. She transferred to Delta State University in Cleveland, Mississippi and received the B.S.E. degree in 1979 with a major in mathematics. She immediately entered graduate school at the same university and received the M. Ed. degree in 1980 with a major in mathematics. In 1978, Tena married William "Skip" Golding. They are the proud parents of William Joshua Golding born in 1981 and Stephen Thomas Golding born in 1984. Upon completion of her master's degree, she became an instructor in the mathematics department at Delta State University and remained there for two years. Since 1982 she has held a position in the mathematics department at Southeastern Louisiana University in Hammond, Louisiana. After moving to Louisiana, she pursued the Doctor of Philosophy degree in Curriculum and Instruction with the emphasis in Mathematics Education awarded in August of 1994 at Louisiana State University.

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Title of Dissertation: The Effects of the Unit Concept on Prospective Elementary Teachers' Understanding of Rational Number Concepts

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